# DSC 40B

## Lecture 24 : Shortest Paths in Weighted Graphs. Bellman-Ford

# Bellman-Ford

# Bellman-Ford

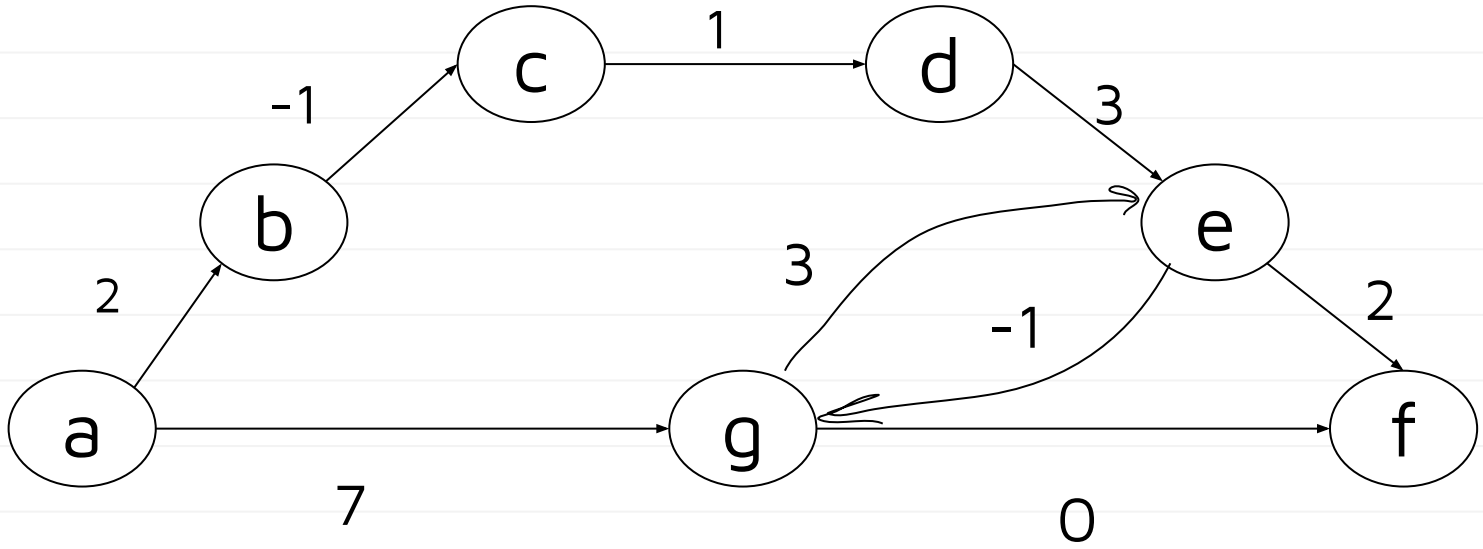
- Dijkstra's algorithm works only with **positive** weights.
- Bellman-Ford allows us to find shortest distances in the graphs with **negative** weights as well.

# Intuition

- Shortest paths that have many edges are “harder” to discover.
  - May require **many** updates.
- Shortest paths that have few edges are “easier” to discover.
- Once we’ve discovered all of the shortest paths with few edges, it makes it easier to discover the shortest paths with more edges.

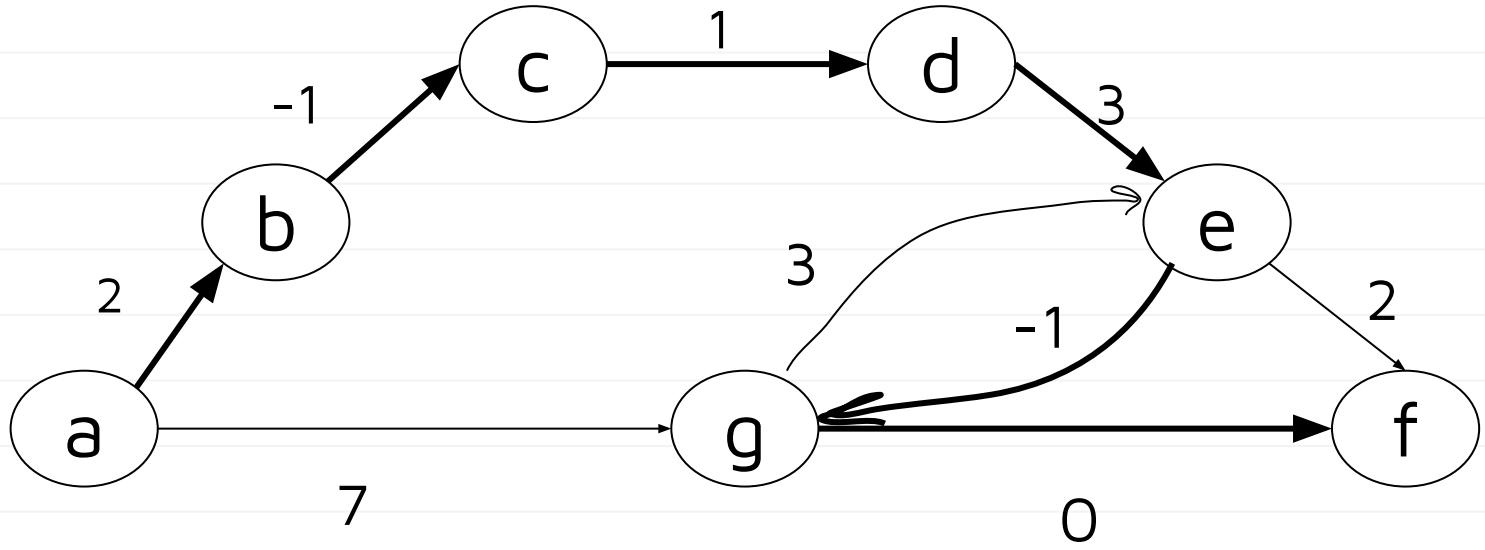
# Example

What is the shortest path from **a** to **f**?



# Example

What is the shortest path from **a** to **f**?

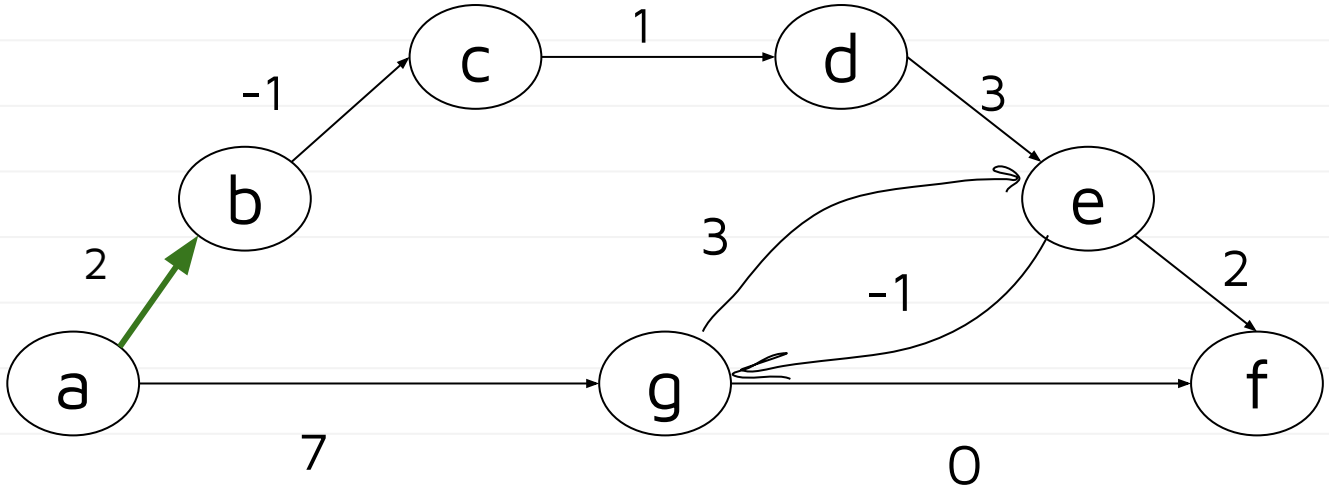



**Path**: a, b, c, d, e, g, f

**Length**: 2 - 1 + 1 + 3 - 1 + 0 = 4

# Updating All Edges

- Suppose we update all of the edges, one by one.
- Then all nodes whose shortest path from $s$ has only **one** edge are **guaranteed** to be estimated correctly.



# Loop Invariant

- **One** iteration: update *all* edges in arbitrary order.
- **Loop invariant**: After $\alpha$ iterations, all nodes whose shortest path from $s$ has $\leq \alpha$ edges are **guaranteed** to be estimated correctly.

# The Bellman-Ford Algorithm

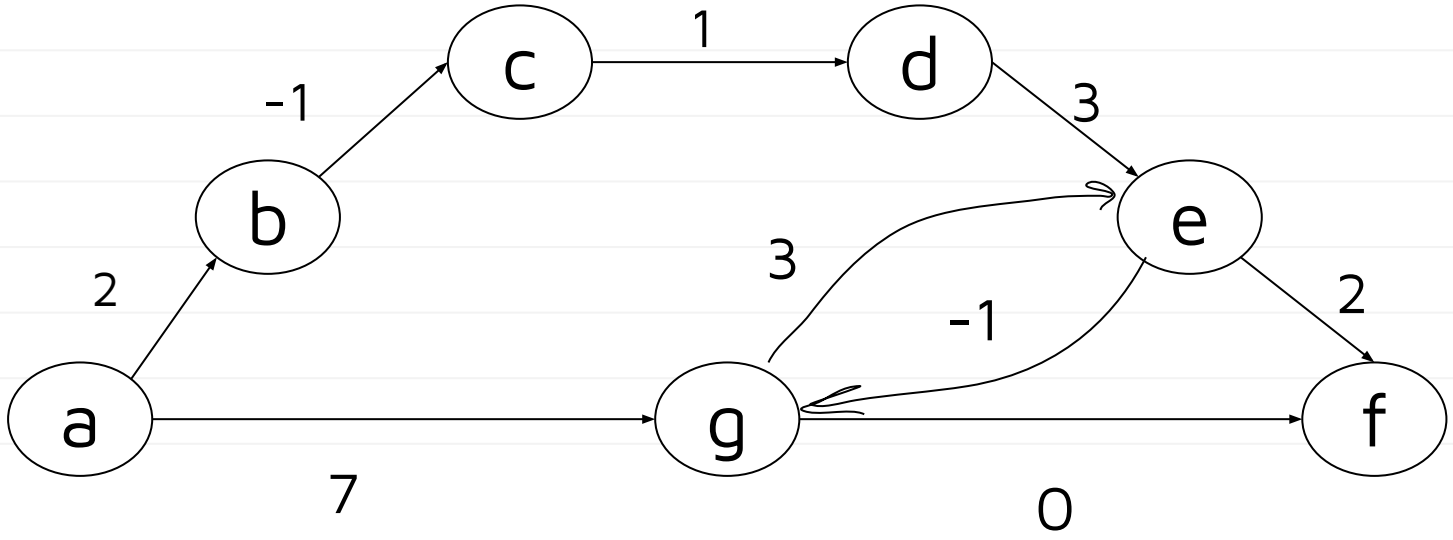
```
def bellman_ford(graph, weights, source):
"""Assume graph is directed."""
    est = {node: float('inf') for node in graph.nodes}
    est[source] = 0
    predecessor = {node: None for node in graph.nodes}

    for i in range(?):
        for (u, v) in graph.edges:
            update(u, v, weights, est, predecessor)
    return est, predecessor
```

Suppose `graph.edges` returns edges in following order:

(c, d), (a, b), (b, c), (g, f), (e, g),

(g, e), (d, e), (e, f), (a, g)



Suppose graph.edges returns edges in following order:

(c, d), (a, b), (b, c), (g, f), (e, g),

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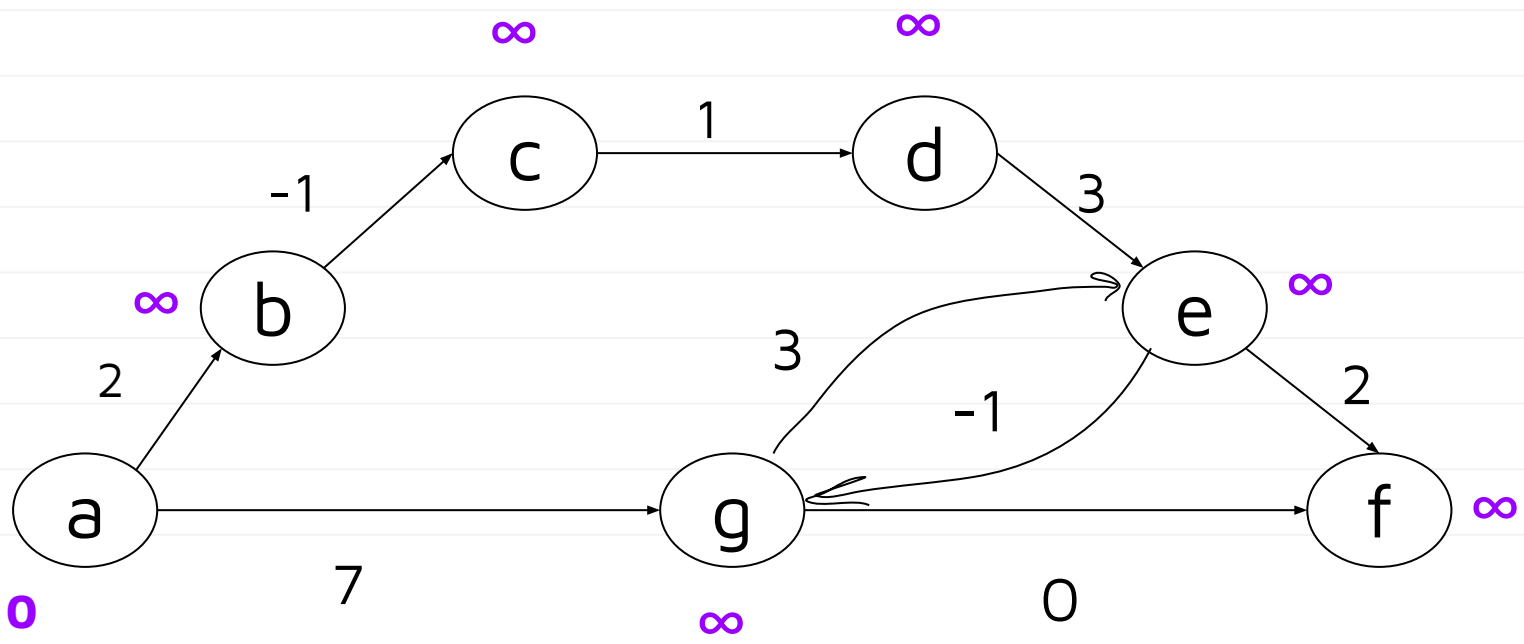
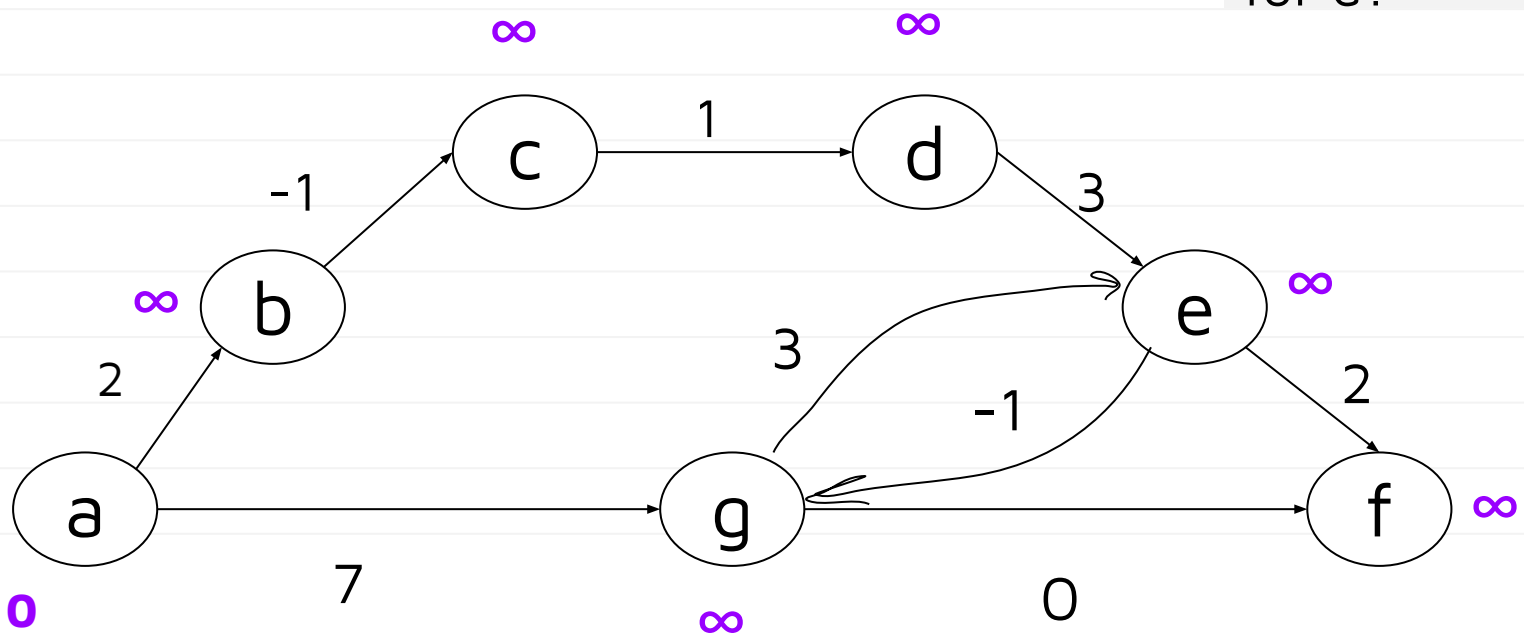


Suppose graph.edges returns edges in following order:

**(c, d)**, (a, b), (b, c), (g, f), (e, g),

(g, e), (d, e), (e, f), (a, g)

Do we update distance for d?

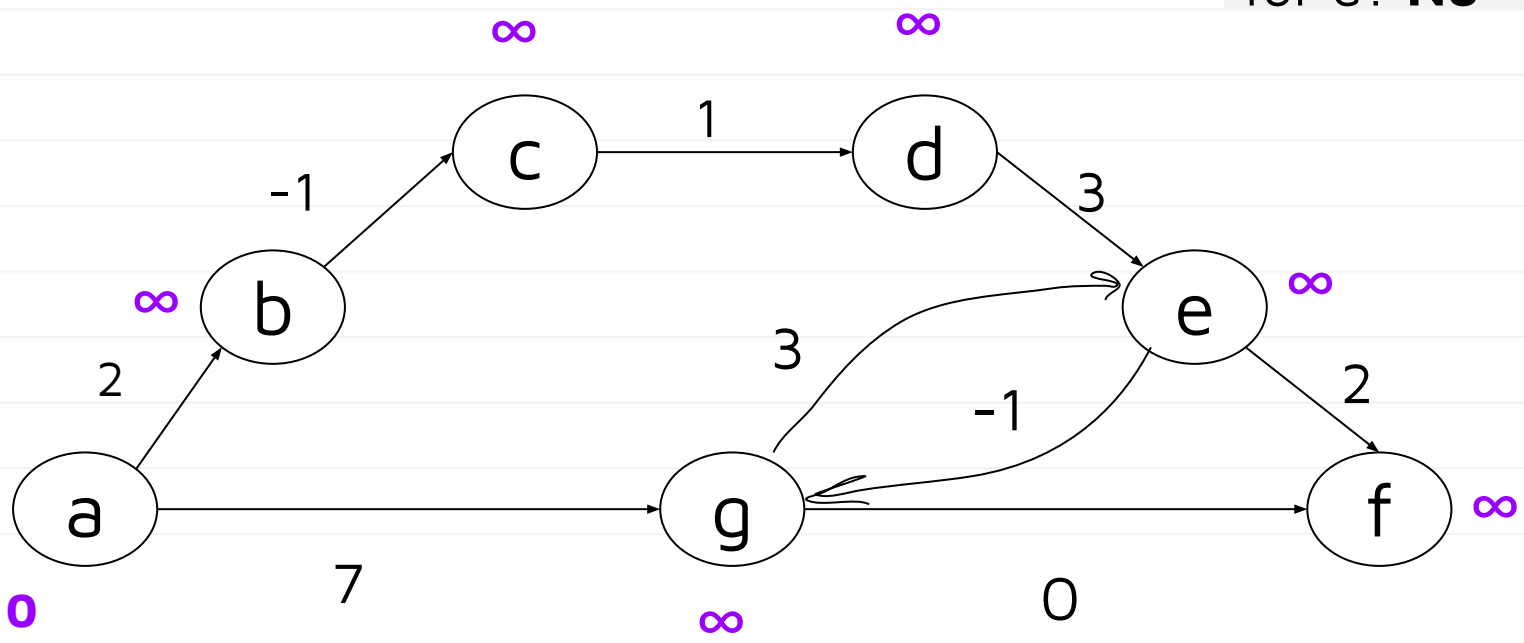


Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, (a, b), (b, c), (g, f), (e, g),

(g, e), (d, e), (e, f), (a, g)

Do we update distance for d? **No**

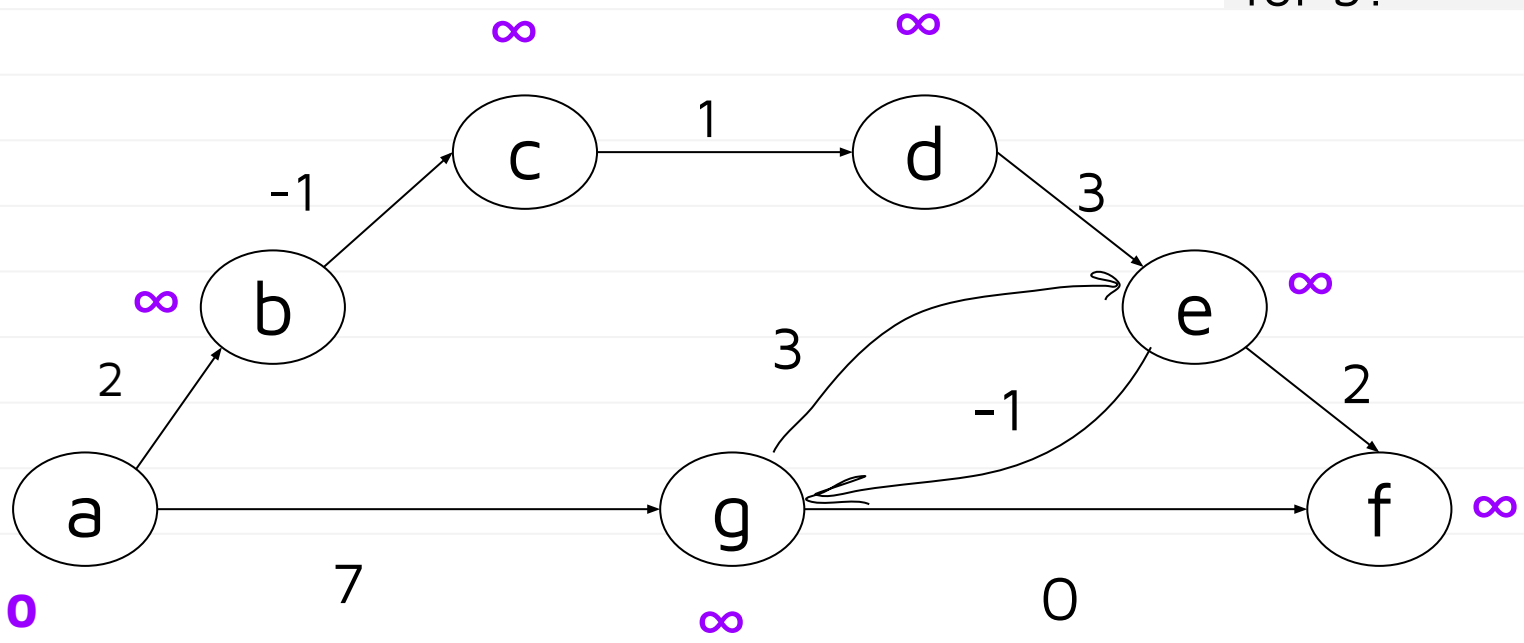


Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, **(a, b)**, (b, c), (g, f), (e, g),

(g, e), (d, e), (e, f), (a, g)

Do we update distance for b?

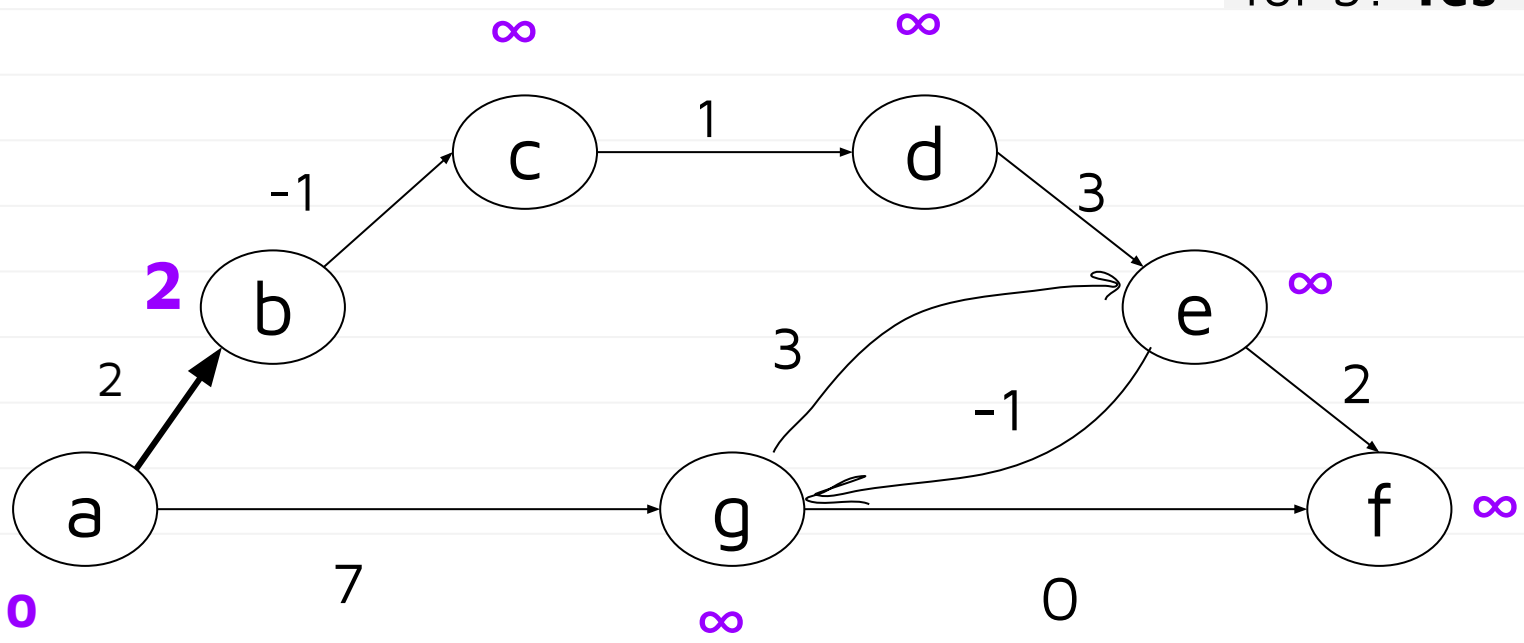


Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, ~~**(a, b)**~~, (b, c), (g, f), (e, g),

(g, e), (d, e), (e, f), (a, g)

Do we update distance for b? **Yes**

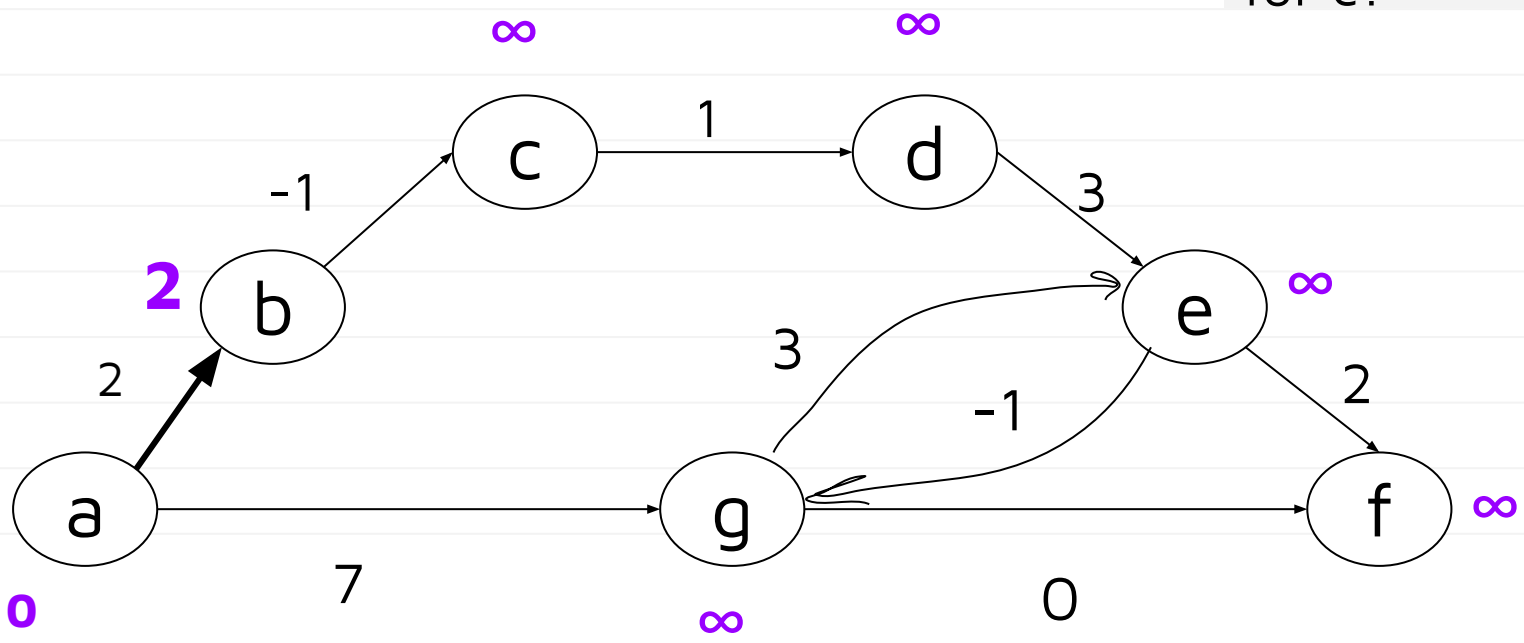


Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, ~~**(a, b)**~~, **(b, c)**, (g, f), (e, g),

(g, e), (d, e), (e, f), (a, g)

Do we update distance for c?

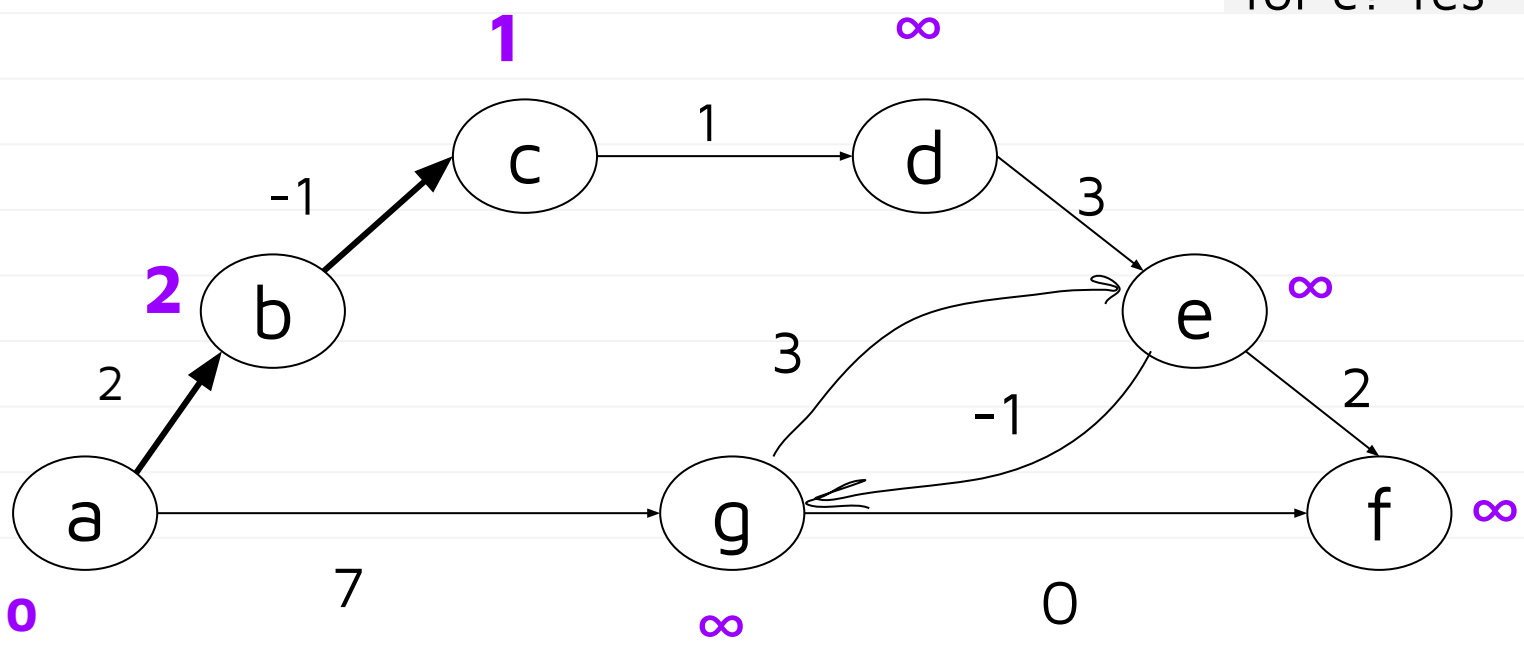


Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, ~~**(a, b)**~~, ~~**(b, c)**~~, (g, f), (e, g),

(g, e), (d, e), (e, f), (a, g)

Do we update distance for c? Yes

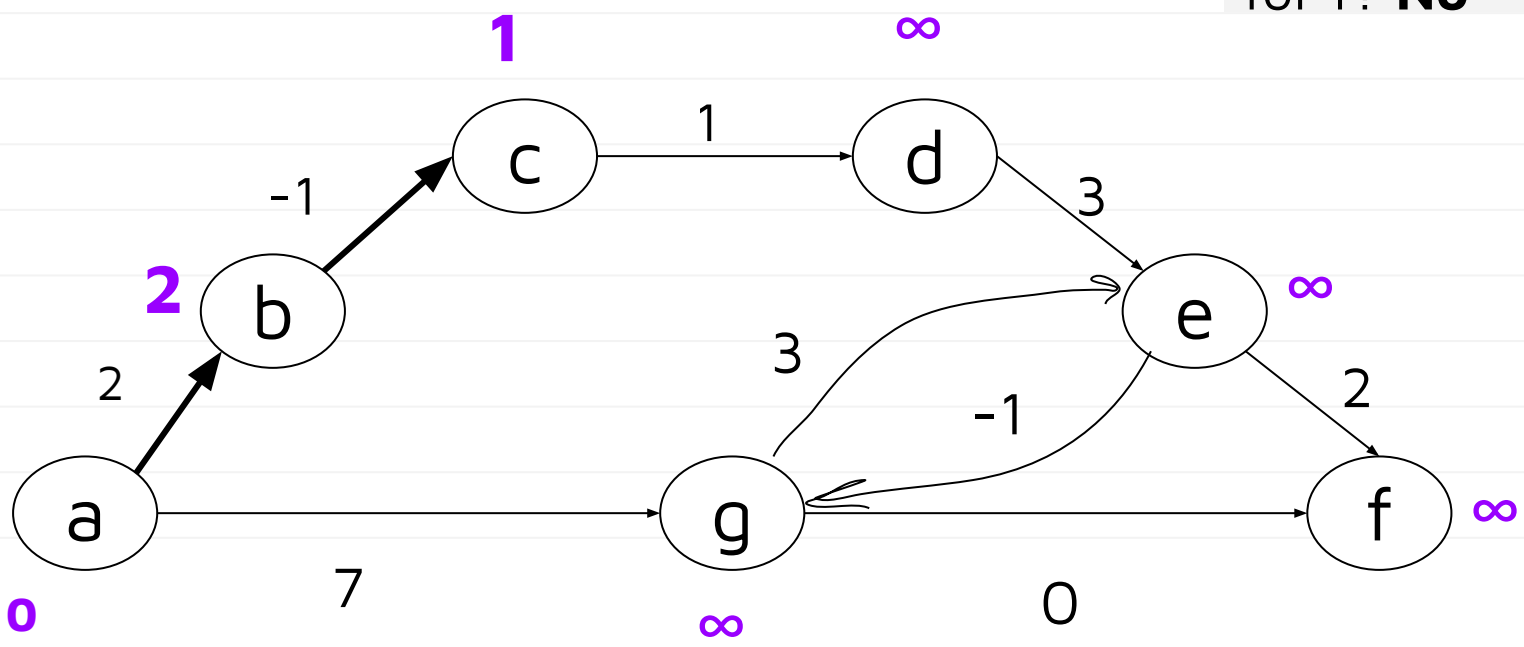


Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, ~~**(a, b)**~~, ~~**(b, c)**~~, **(g, f)**, (e, g),

(g, e), (d, e), (e, f), (a, g)

Do we update distance for f? **No**

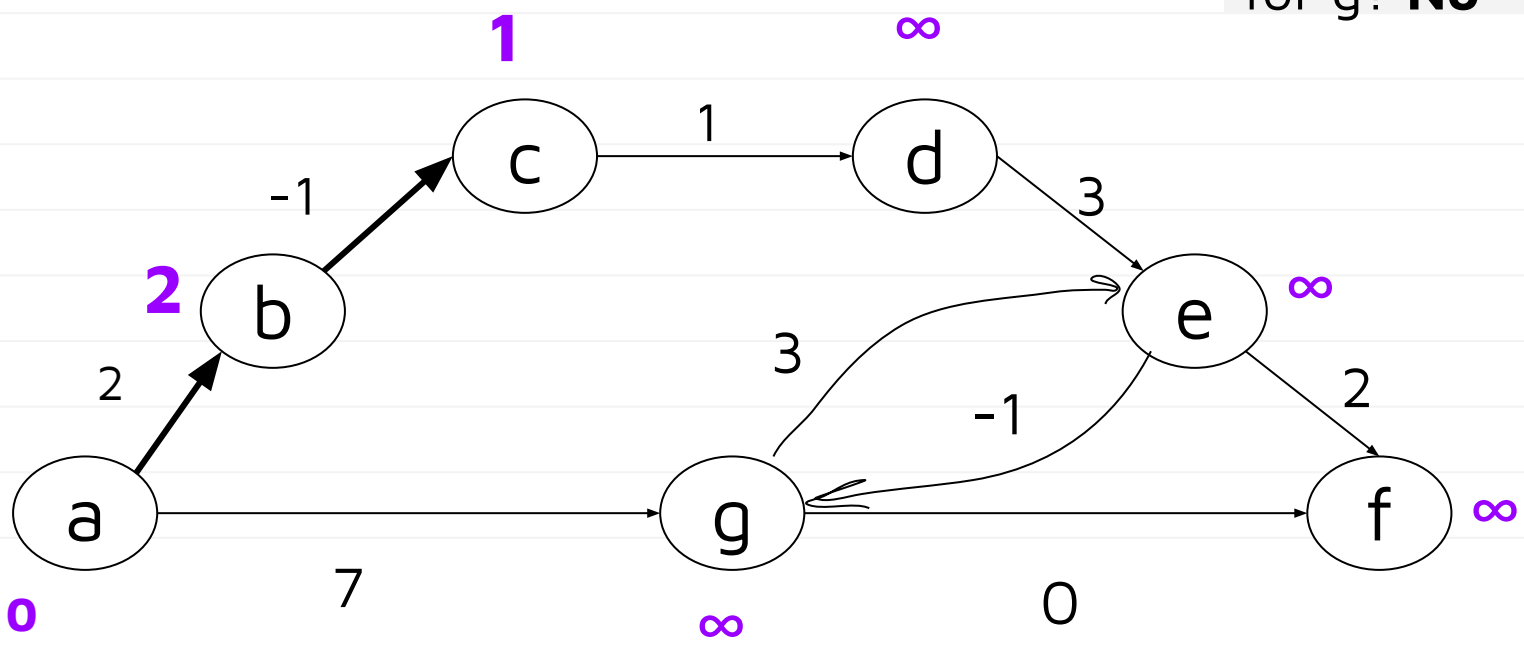


Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, ~~**(a, b)**~~, ~~**(b, c)**~~, ~~**(g, f)**~~, **(e, g)**,

(g, e), (d, e), (e, f), (a, g)

Do we update distance for g? **No**

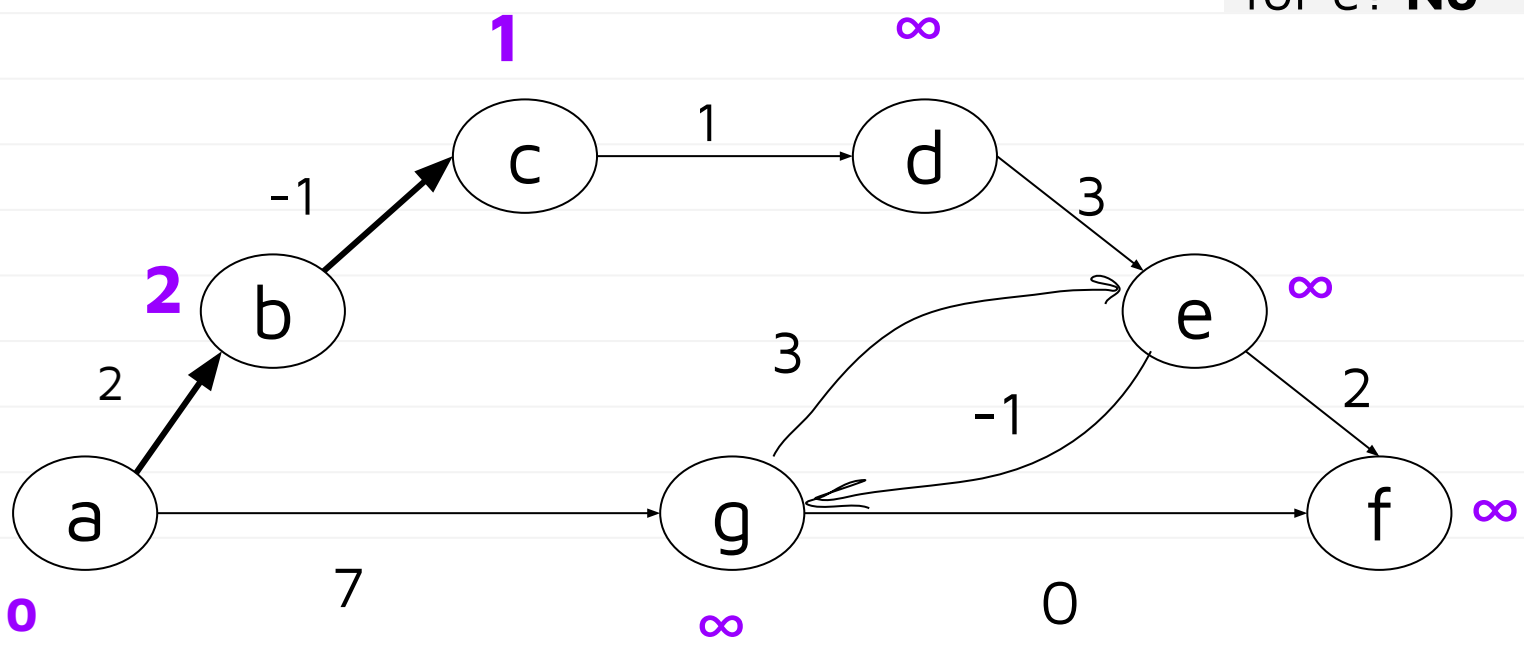


Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, ~~**(a, b)**~~, ~~**(b, c)**~~, ~~**(g, f)**~~, ~~**(e, g)**~~,

**(g, e)**, (d, e), (e, f), (a, g)

Do we update distance for e? **No**

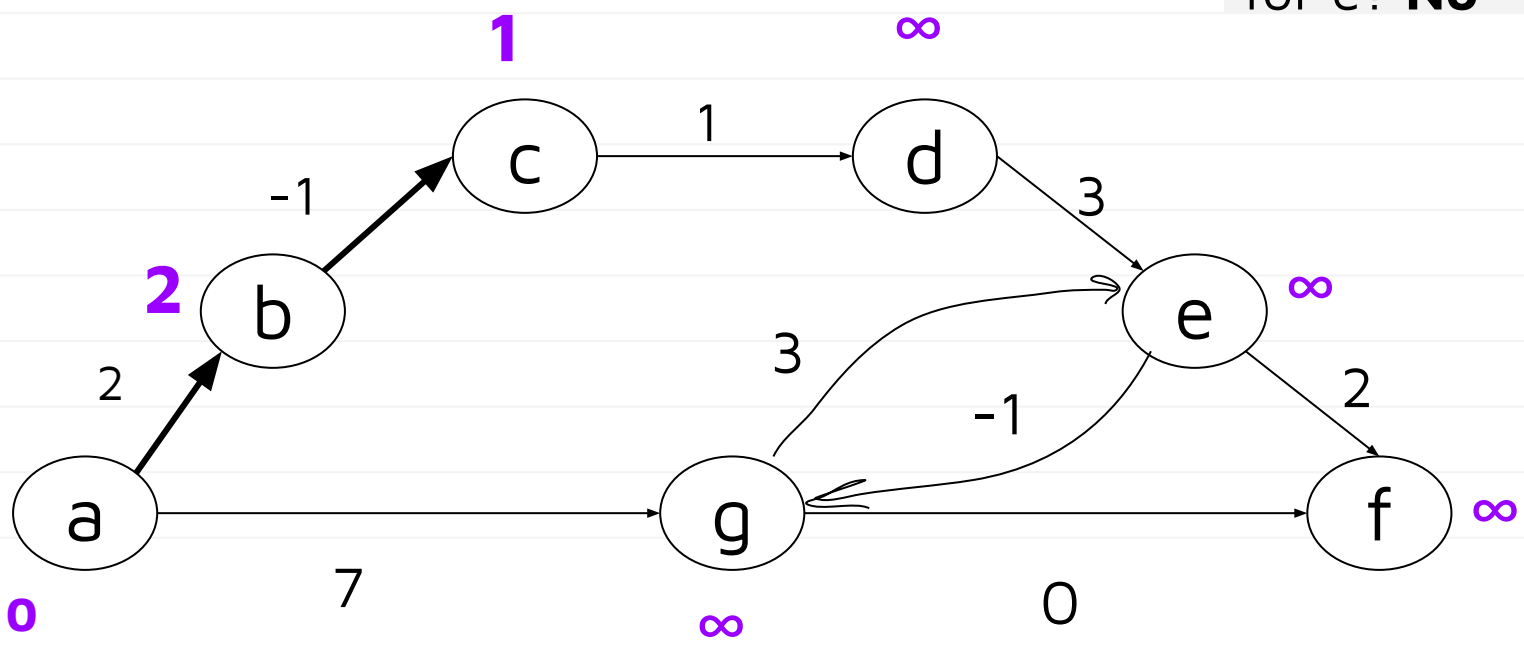


Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, ~~**(a, b)**~~, ~~**(b, c)**~~, ~~**(g, f)**~~, ~~**(e, g)**~~,

~~**(g, e)**~~, **(d, e)**, (e, f), (a, g)

Do we update distance for e? **No**

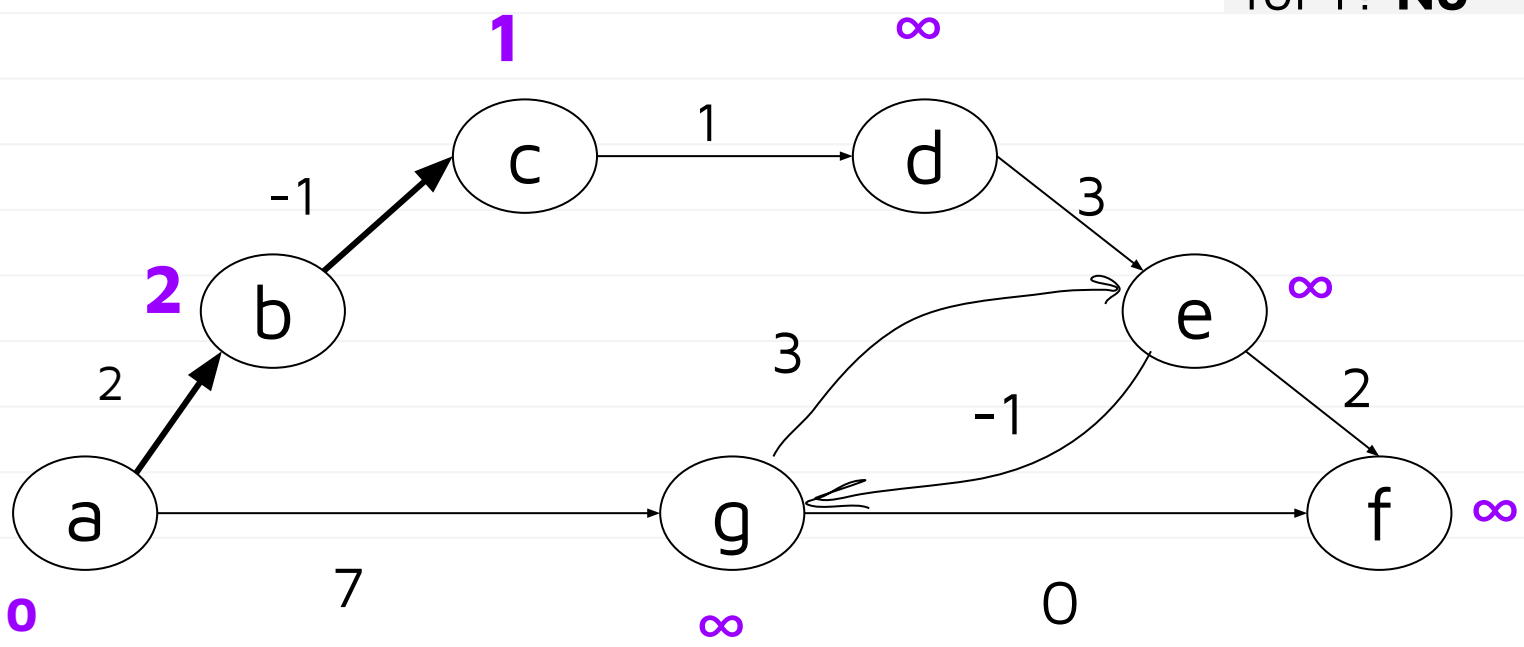


Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, ~~**(a, b)**~~, ~~**(b, c)**~~, ~~**(g, f)**~~, ~~**(e, g)**~~,

~~**(g, e)**~~, ~~**(d, e)**~~, **(e, f)**, (a, g)

Do we update distance for f? **No**

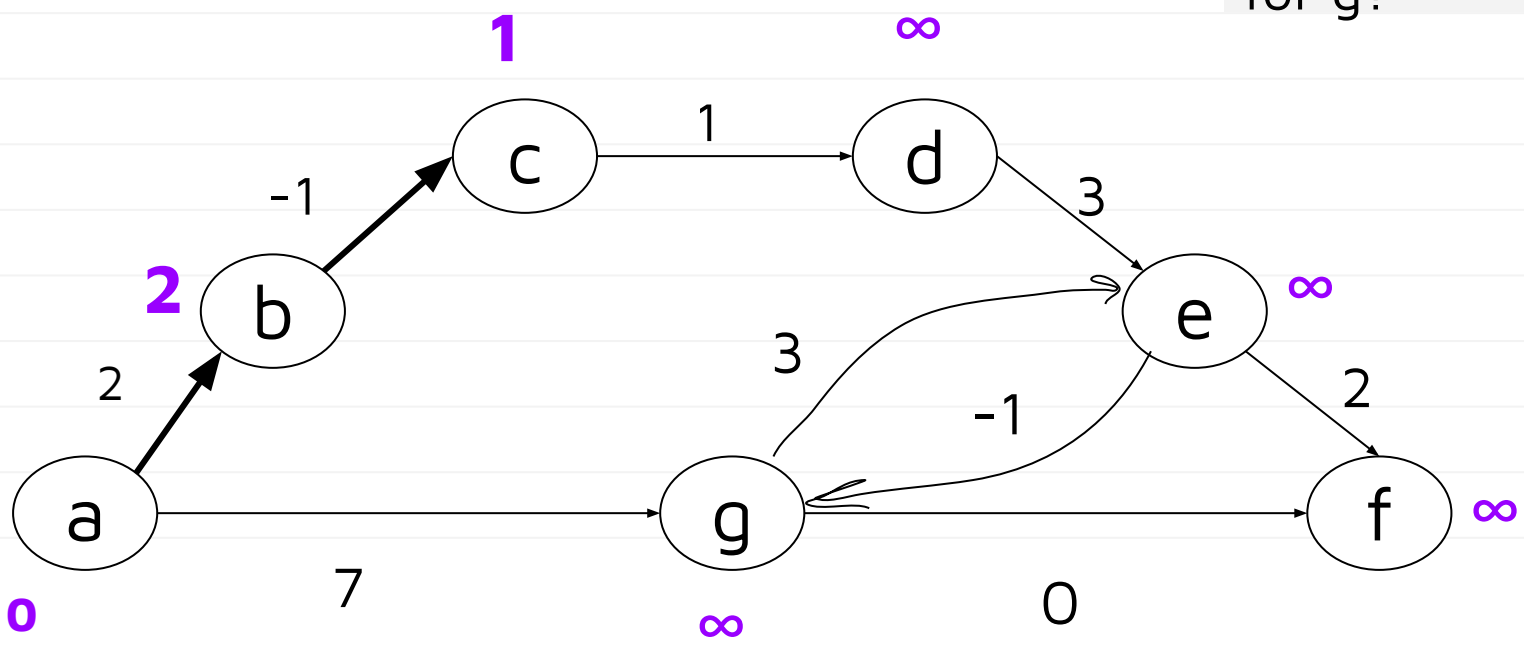


Suppose graph.edges returns edges in following order:

**~~(c, d)~~**, **~~(a, b)~~**, **~~(b, c)~~**, **~~(g, f)~~**, **~~(e, g)~~**,

**~~(g, e)~~**, **~~(d, e)~~**, **~~(e, f)~~**, **(a, g)**

Do we update distance for g?

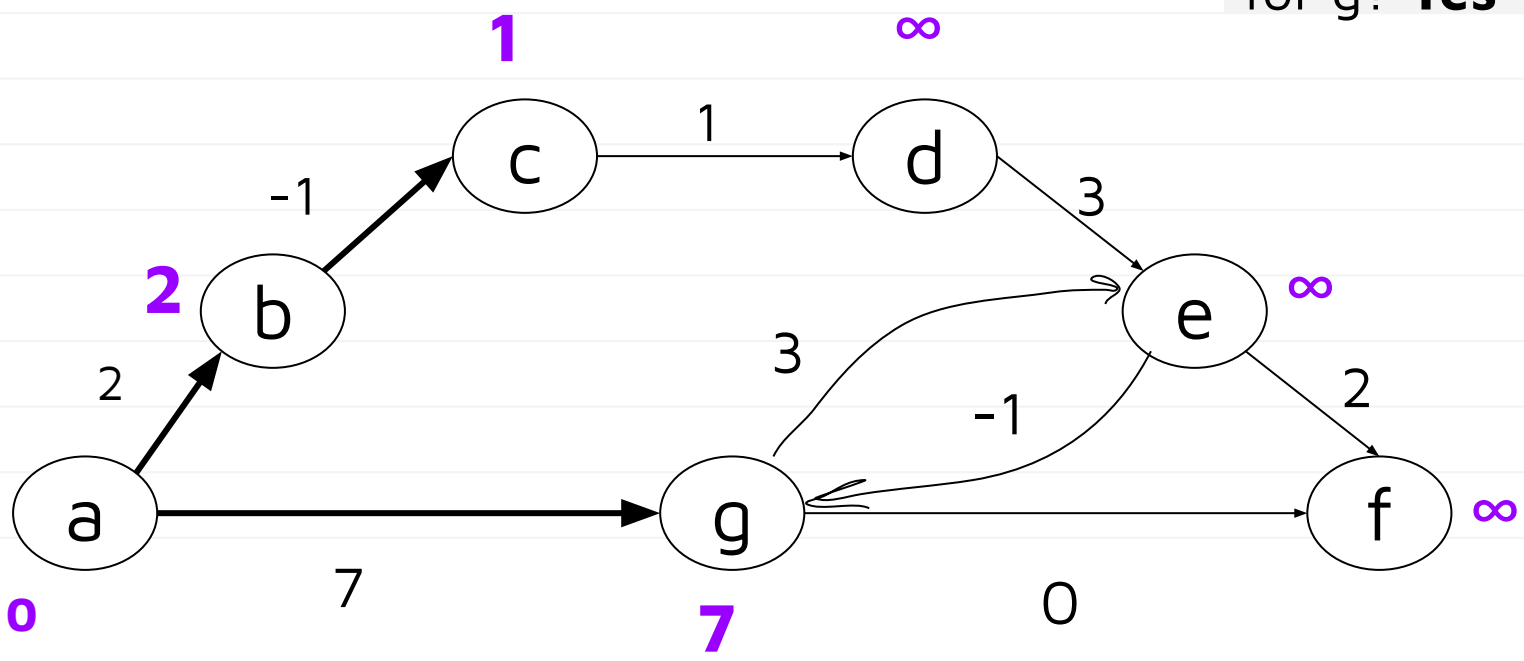


Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, ~~**(a, b)**~~, ~~**(b, c)**~~, ~~**(g, f)**~~, ~~**(e, g)**~~,

~~**(g, e)**~~, ~~**(d, e)**~~, ~~**(e, f)**~~, ~~**(a, g)**~~

Do we update distance for g? **Yes**

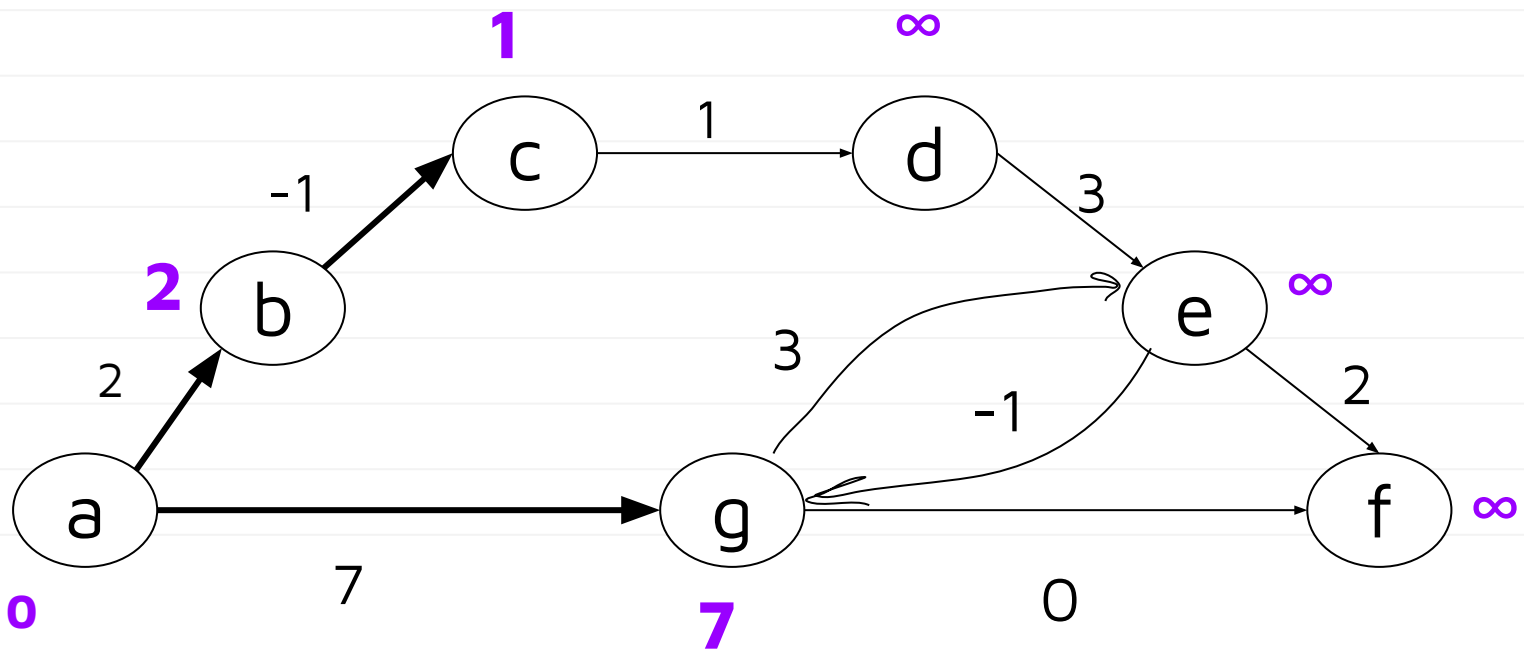


Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, ~~**(a, b)**~~, ~~**(b, c)**~~, ~~**(g, f)**~~, ~~**(e, g)**~~,

~~**(g, e)**~~, ~~**(d, e)**~~, ~~**(e, f)**~~, ~~**(a, g)**~~

It was just a **single** iteration of BF

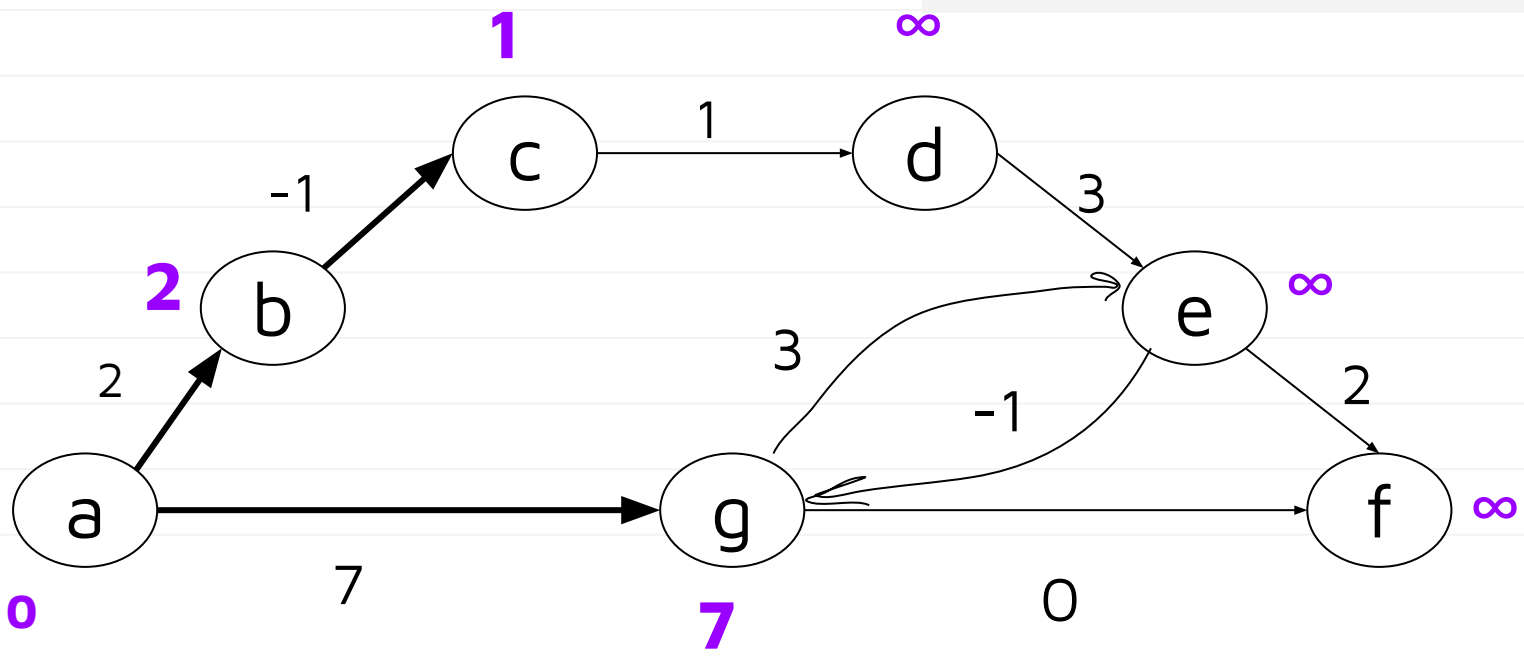


Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, ~~**(a, b)**~~, ~~**(b, c)**~~, ~~**(g, f)**~~, ~~**(e, g)**~~,

~~**(g, e)**~~, ~~**(d, e)**~~, ~~**(e, f)**~~, ~~**(a, g)**~~

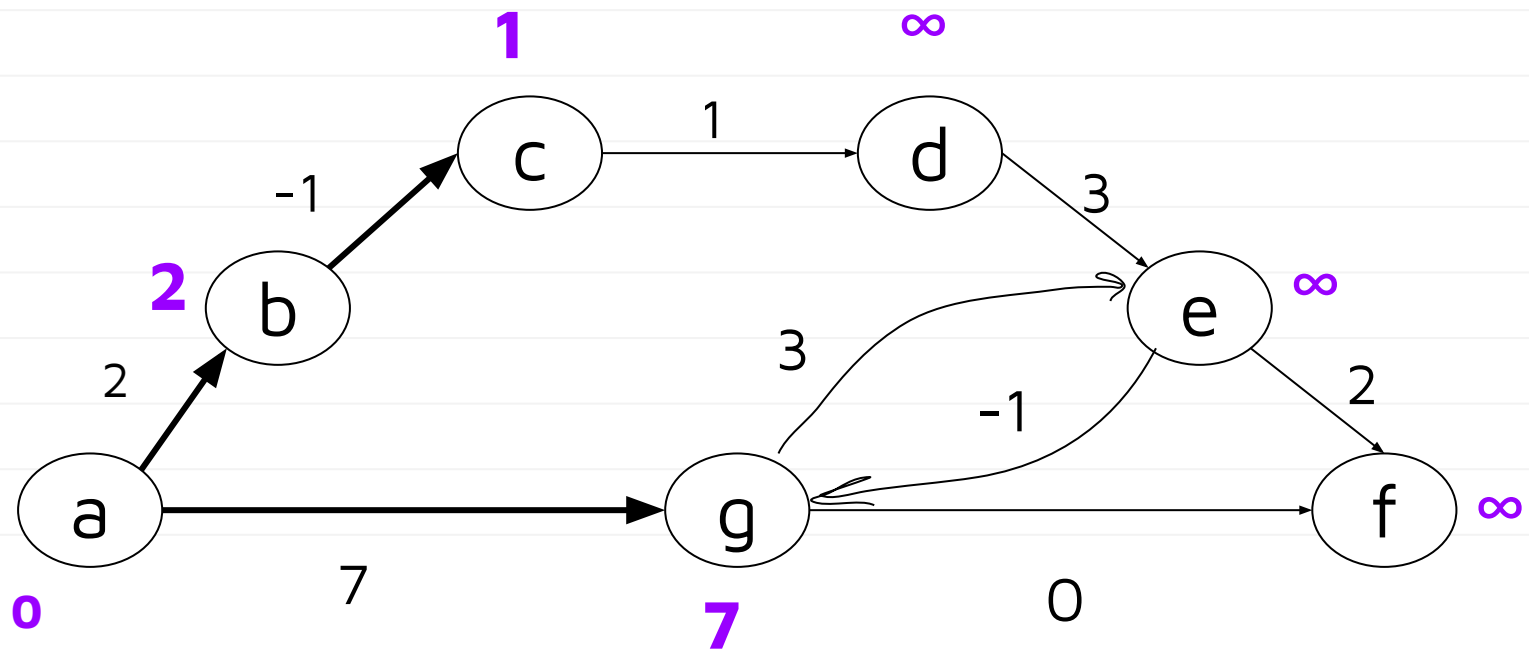
We only can guarantee one correct edge after this iteration: (a, b)



Suppose graph.edges returns edges in following order:

**(c, d)**, (a, b), (b, c), (g, f), (e, g), (g, e), (d, e), (e, f), (a, g)

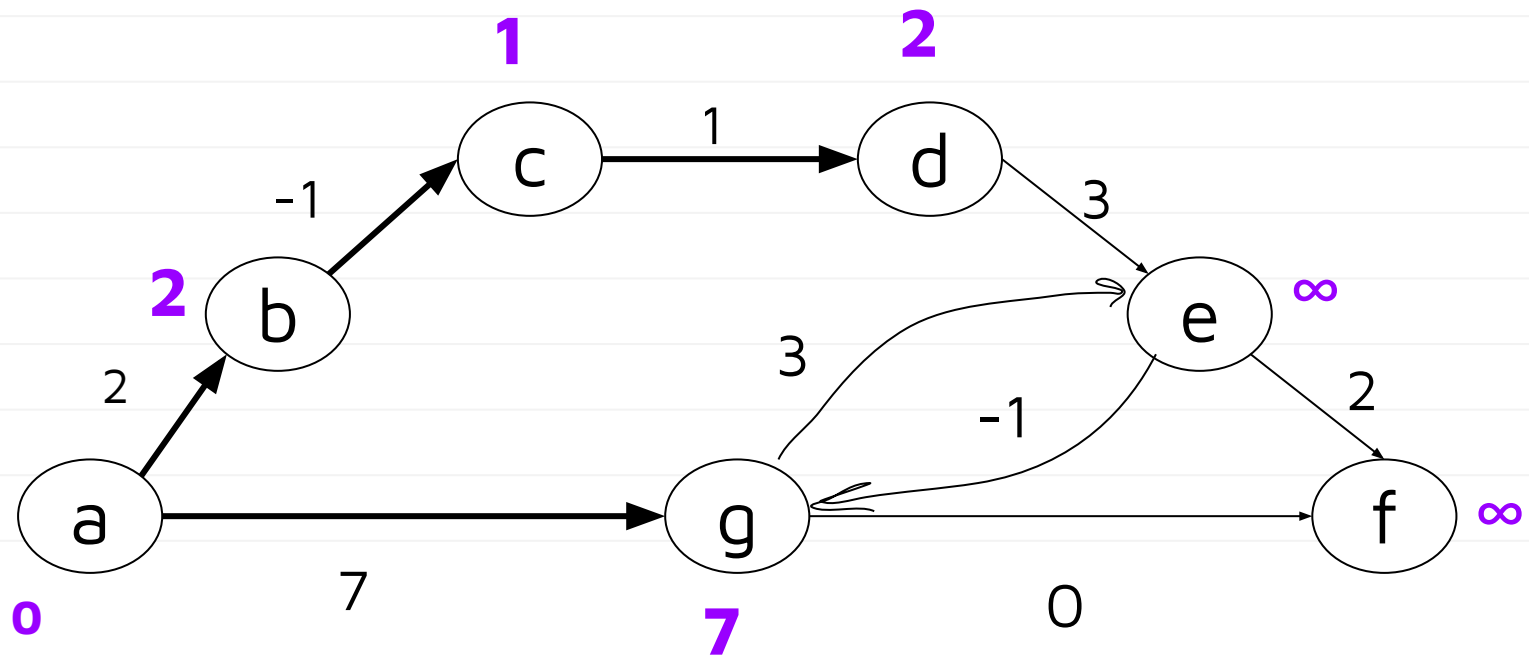
Do we update distance for d? **Yes**



Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, **(a, b)**, (b, c), (g, f), (e, g), (g, e), (d, e), (e, f), (a, g)

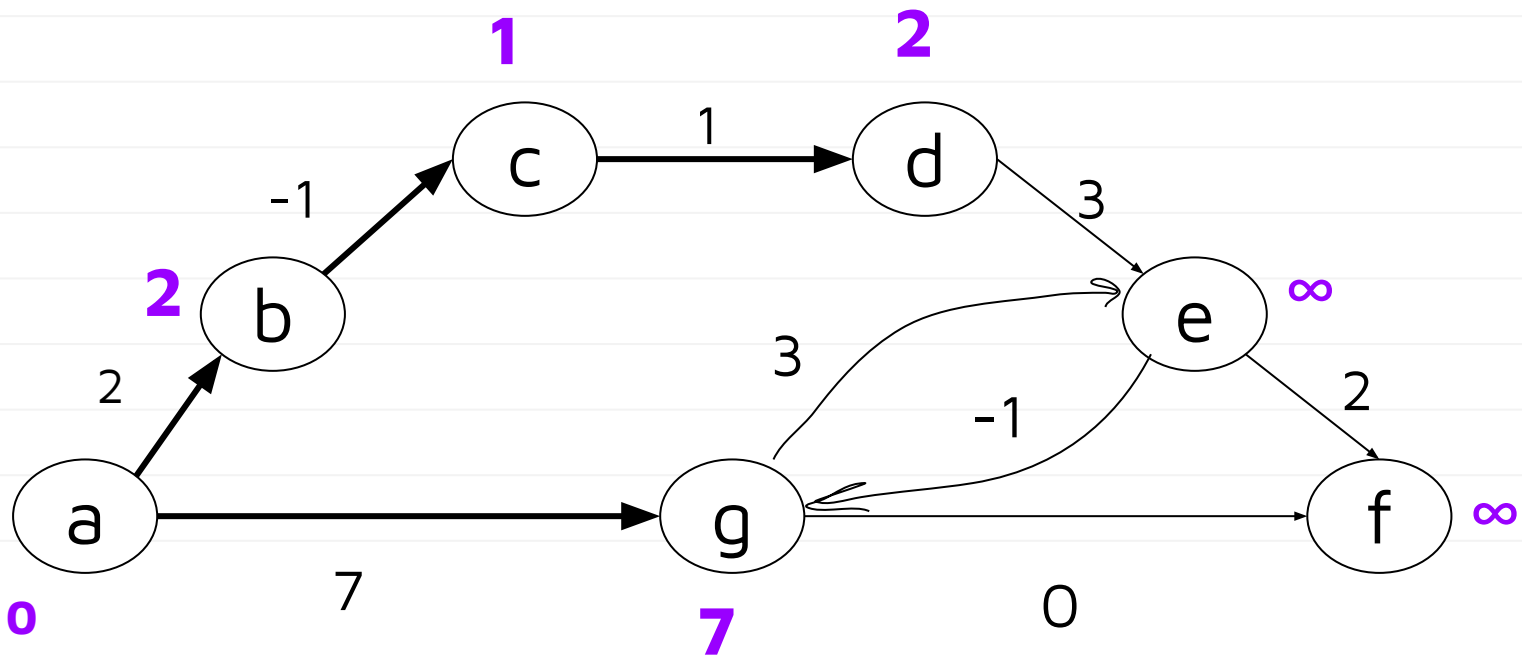
Do we update distance for b? **No**



Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, ~~**(a, b)**~~, **(b, c)**, (g, f), (e, g), (g, e), (d, e), (e, f), (a, g)

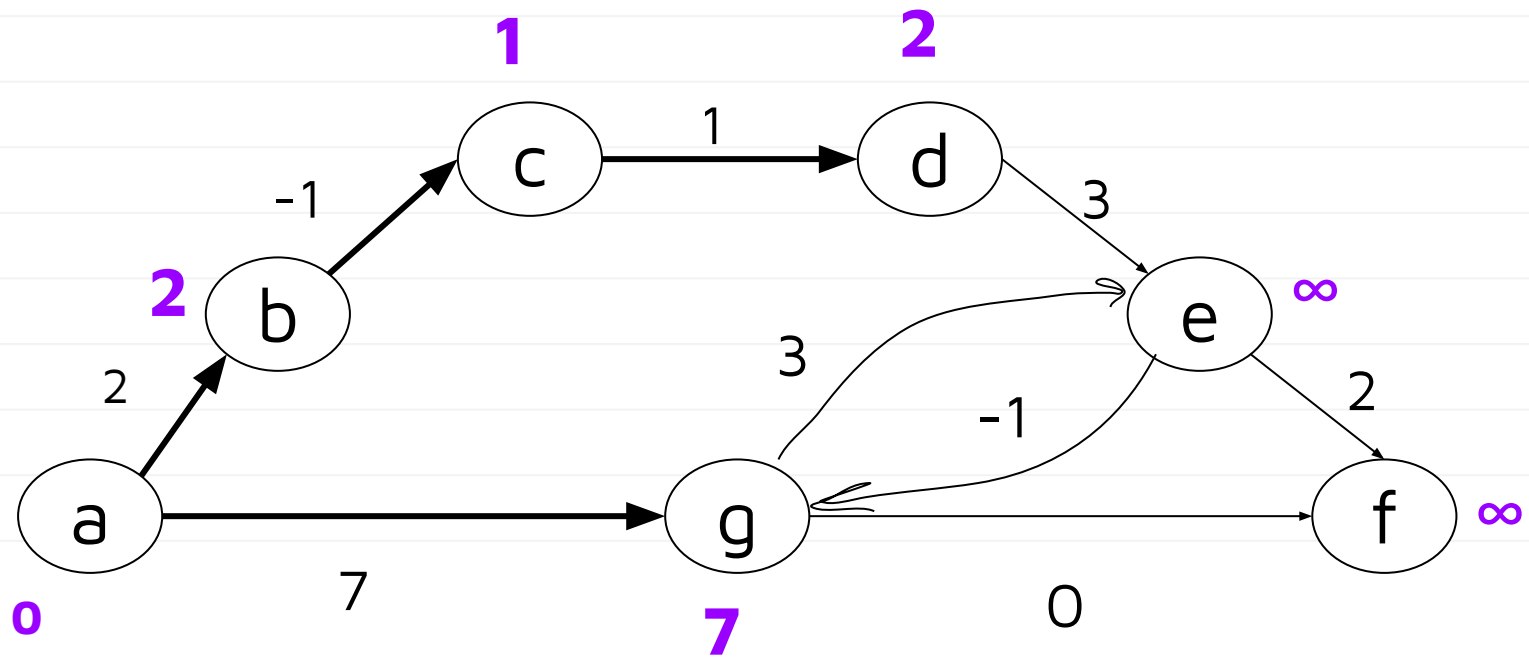
Do we update distance for c? **No**



Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, ~~**(a, b)**~~, ~~**(b, c)**~~, **(g, f)**, (e, g), (g, e), (d, e), (e, f), (a, g)

Do we update distance for f? **Yes**



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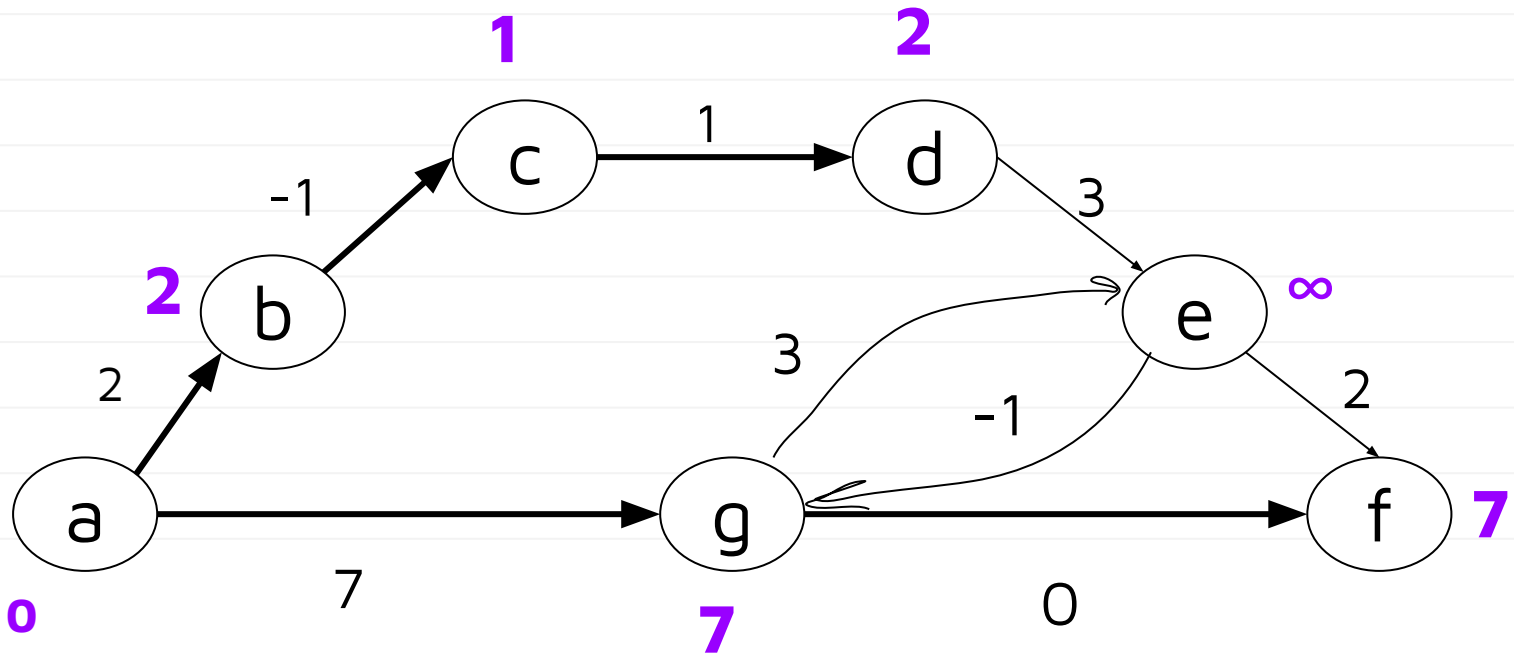
~~**(c, d)**~~, ~~**(a, b)**~~, ~~**(b, c)**~~, **(g, f)**, (e, g), (g, e), (d, e), (e, f), (a, g)

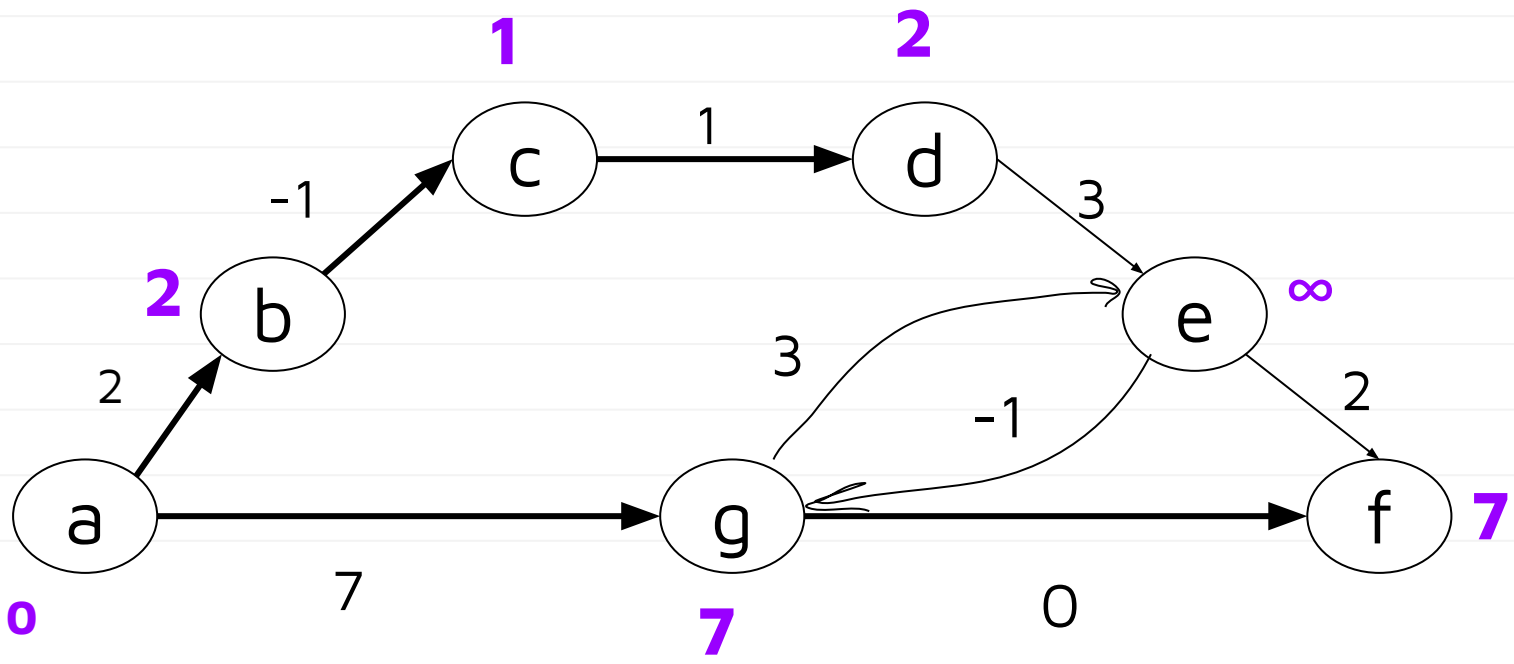
Do we update distance for f? **Yes**



Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, ~~**(a, b)**~~, ~~**(b, c)**~~, ~~**(g, f)**~~, **(e, g)**, (g, e), (d, e), (e, f), (a, g)

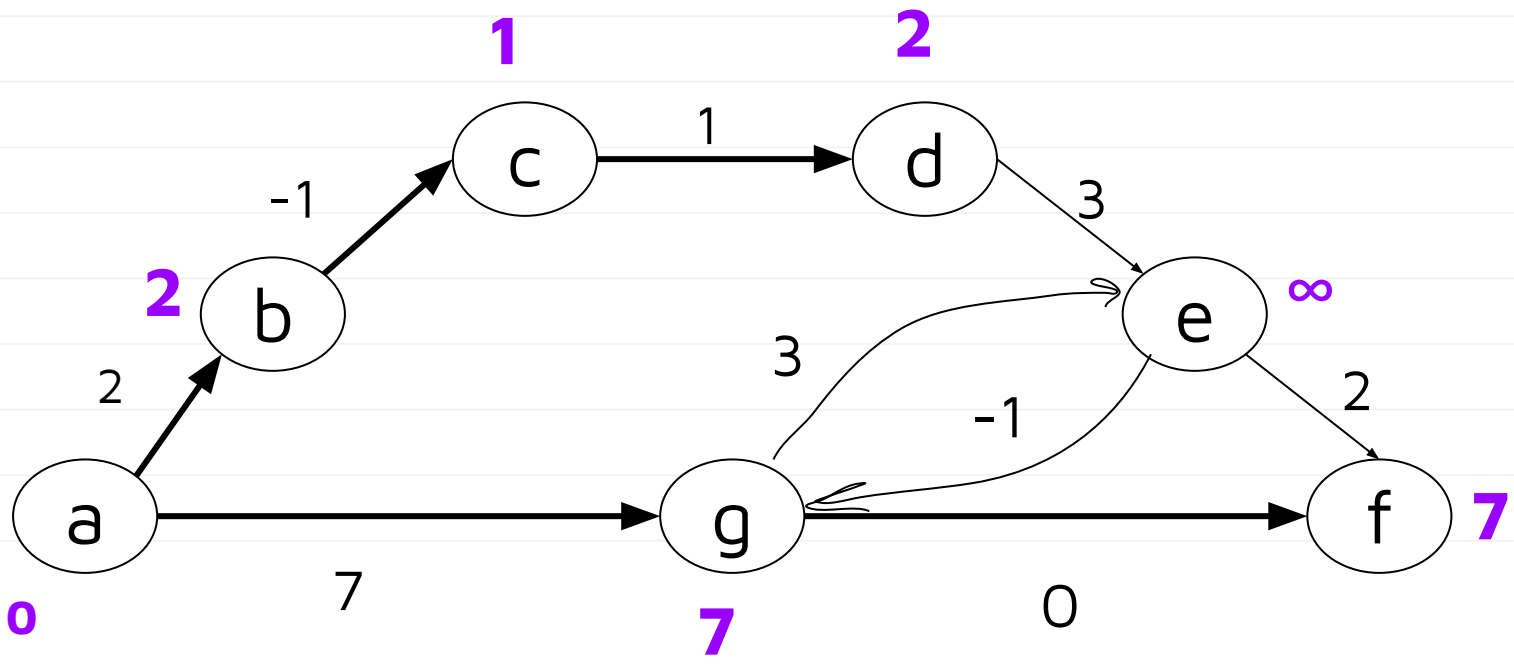
Do we update distance for g? **No**



Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, ~~**(a, b)**~~, ~~**(b, c)**~~, ~~**(g, f)**~~, ~~**(e, g)**~~, **(g, e)**, (d, e), (e, f), (a, g)

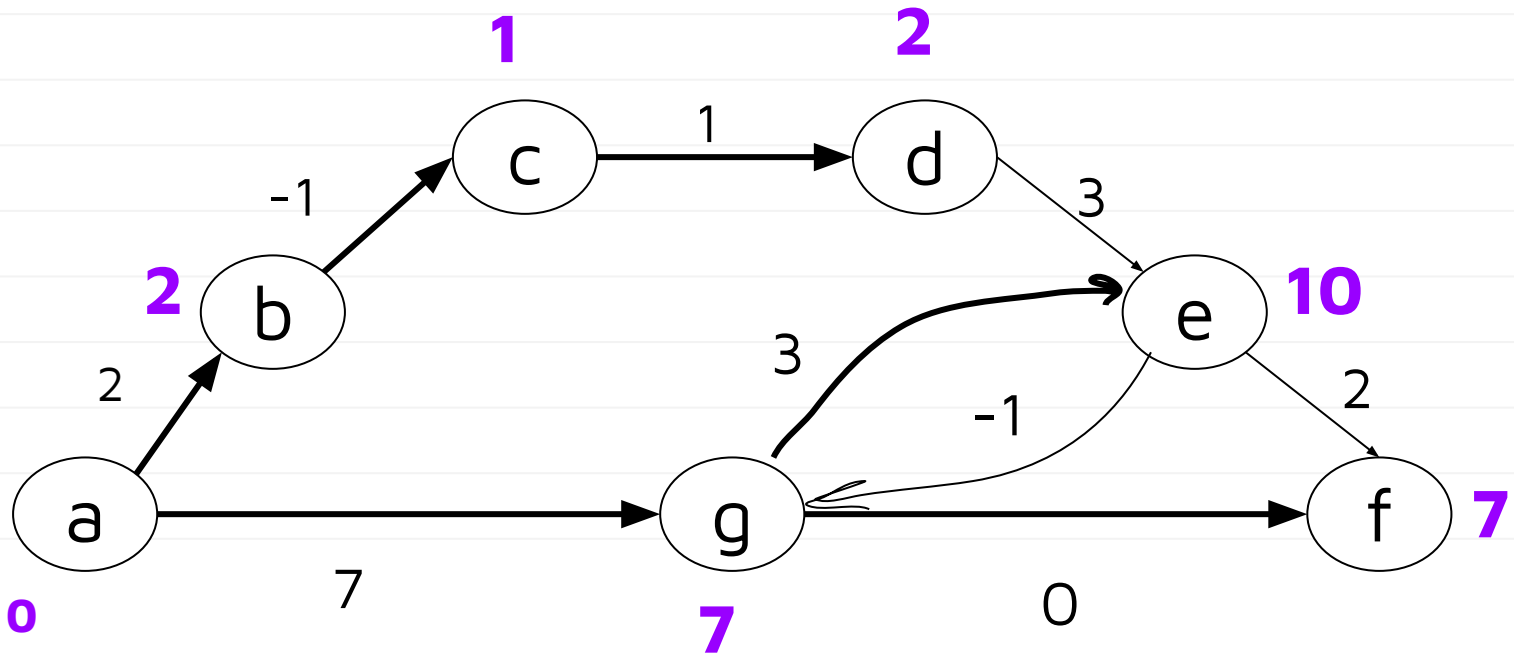
Do we update distance for e? **Yes**



Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, ~~**(a, b)**~~, ~~**(b, c)**~~, ~~**(g, f)**~~, ~~**(e, g)**~~, ~~**(g, e)**~~, (d, e), (e, f), (a, g)

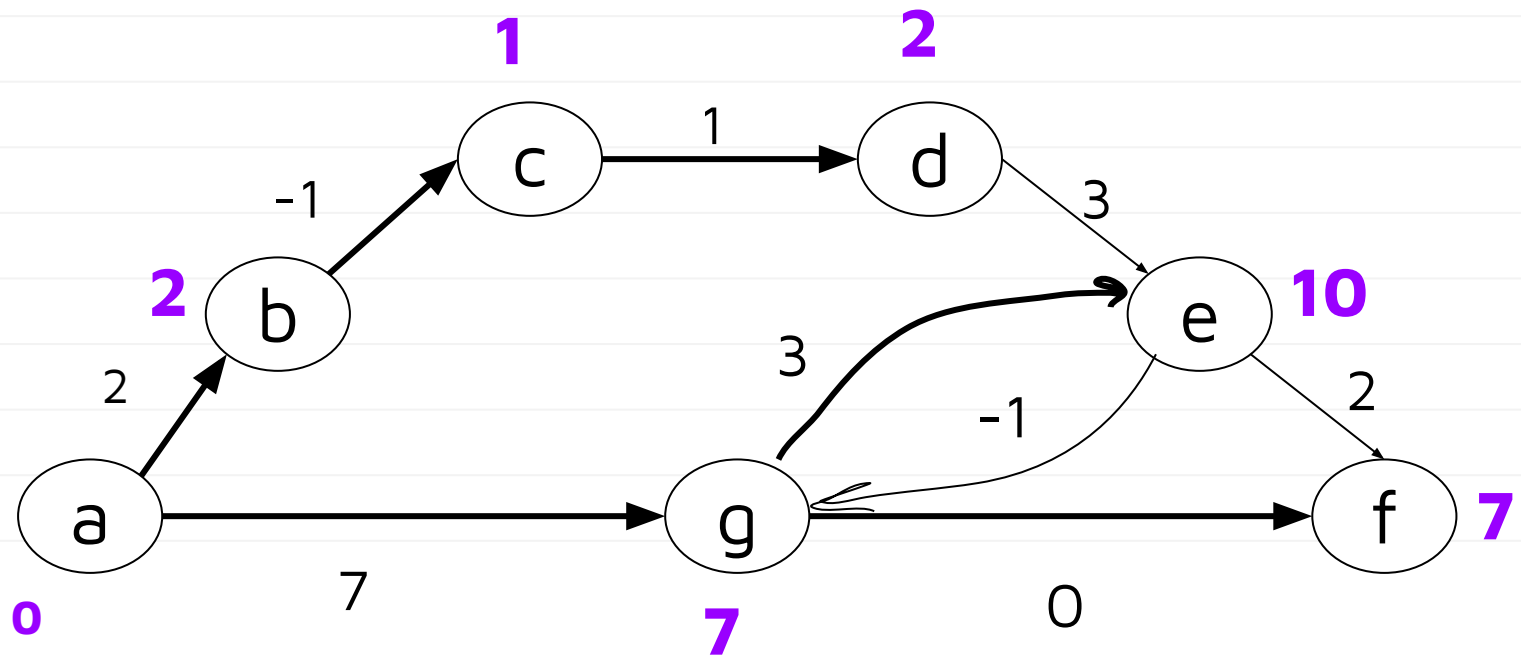
Do we update distance for e? **Yes**



Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, ~~**(a, b)**~~, ~~**(b, c)**~~, ~~**(g, f)**~~, ~~**(e, g)**~~, ~~**(g, e)**~~, **(d, e)**, (e, f), (a, g)

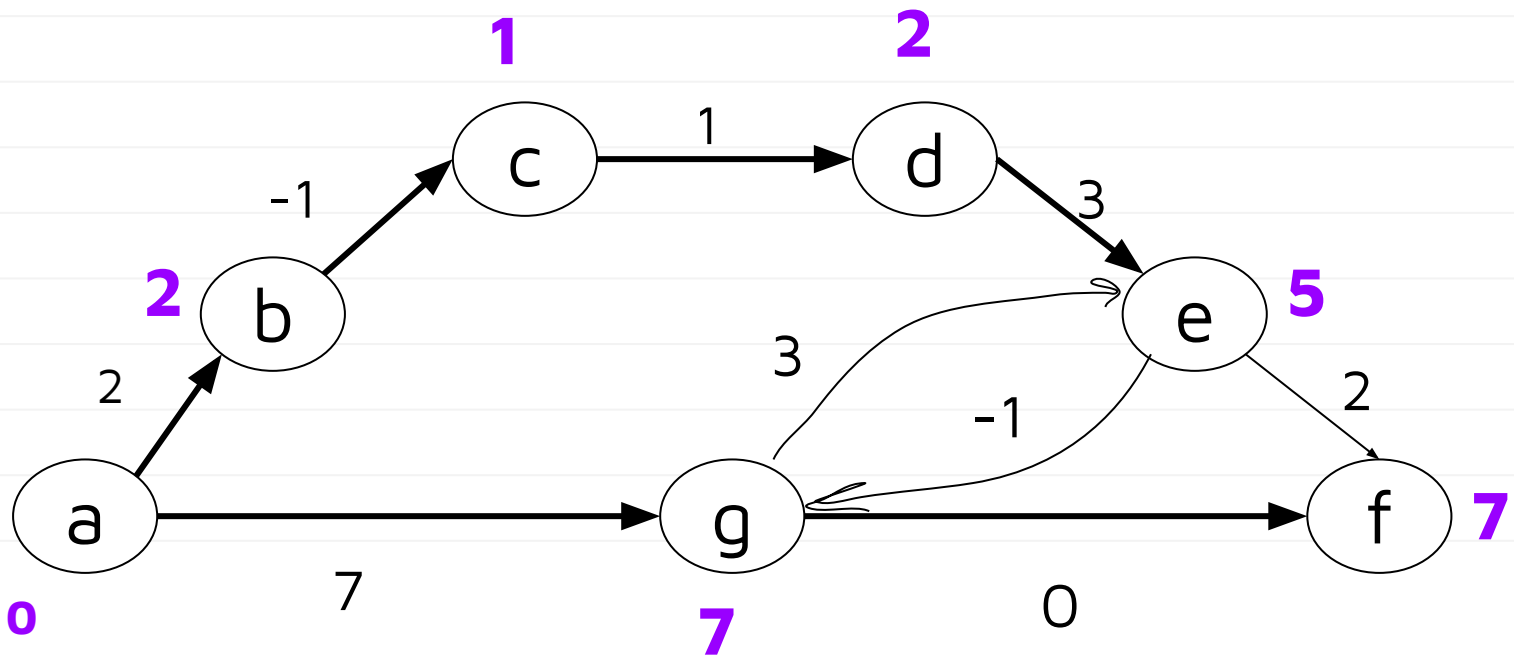
Do we update distance for e? **Yes**



Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, ~~**(a, b)**~~, ~~**(b, c)**~~, ~~**(g, f)**~~, ~~**(e, g)**~~, ~~**(g, e)**~~, ~~**(d, e)**~~, **(e, f)**, (a, g)

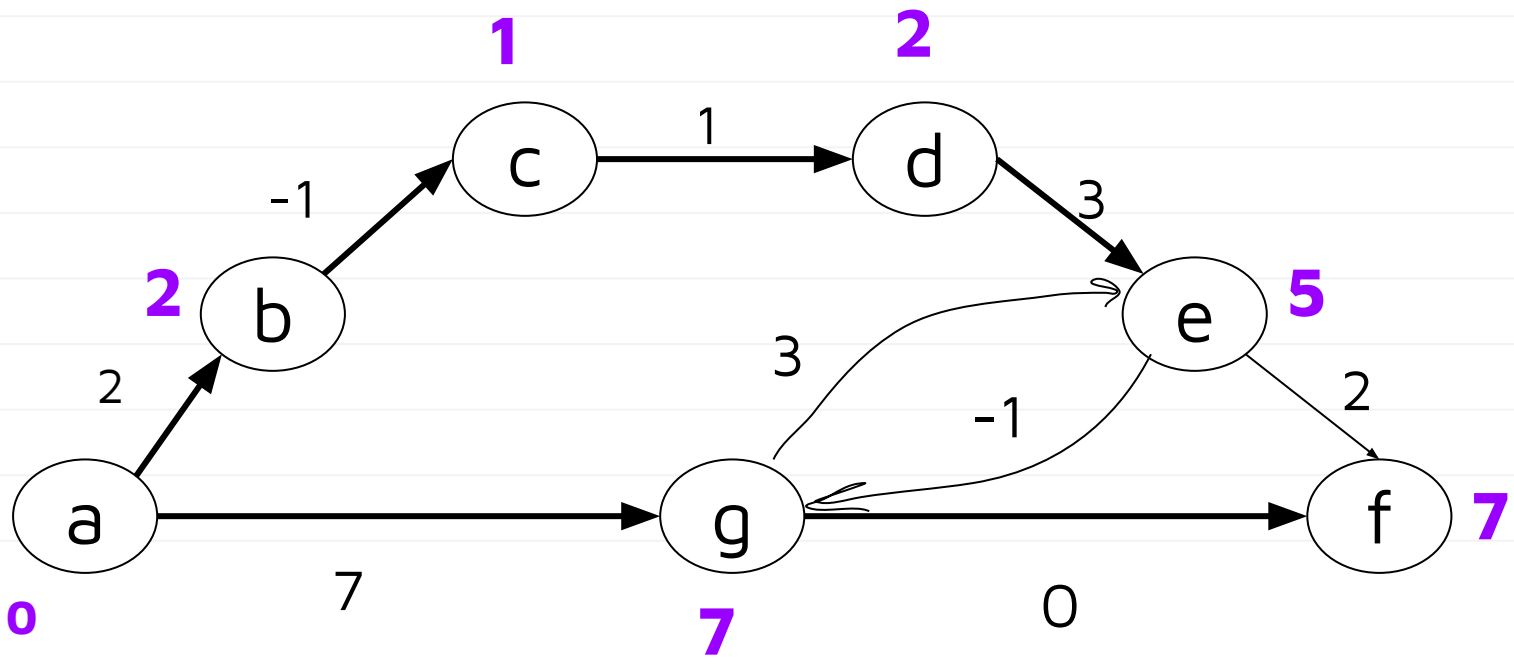
Do we update distance for f? **No**



Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, ~~**(a, b)**~~, ~~**(b, c)**~~, ~~**(g, f)**~~, ~~**(e, g)**~~, ~~**(g, e)**~~, ~~**(d, e)**~~, ~~**(e, f)**~~, **(a, g)**

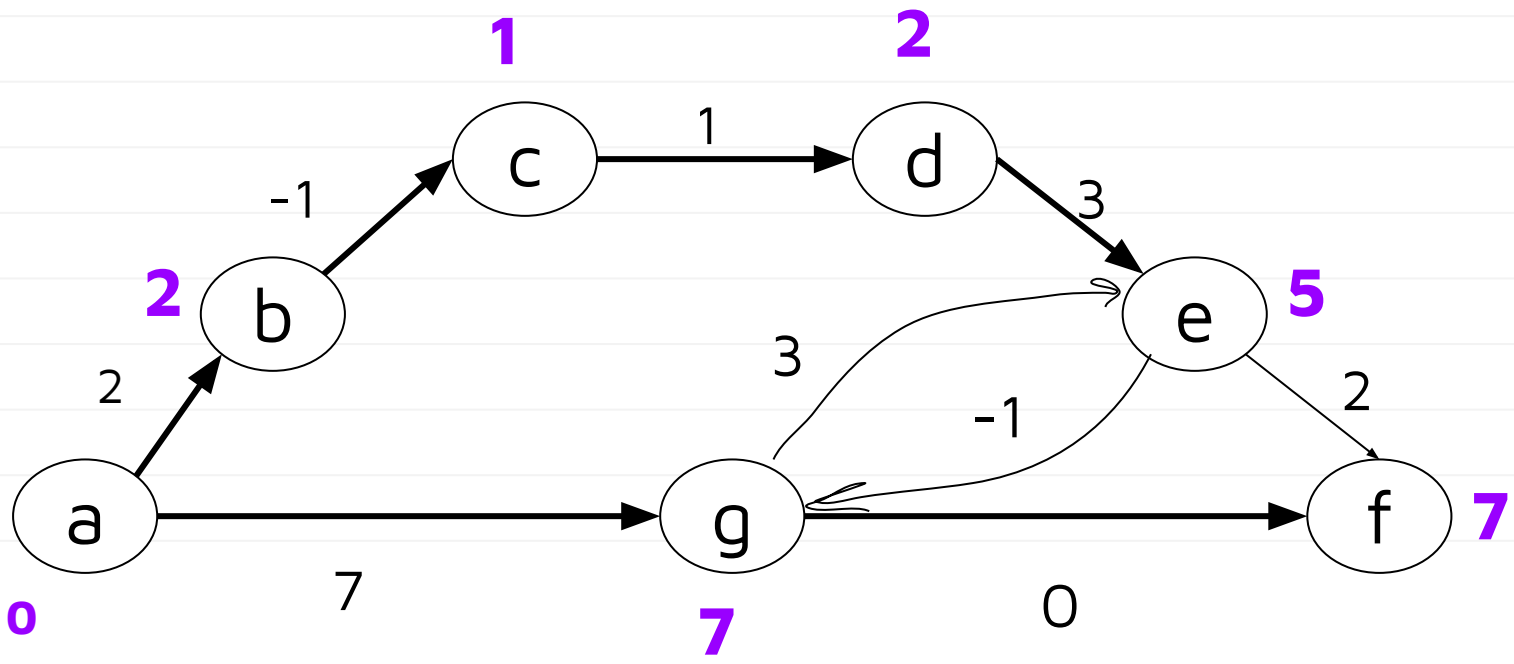
Do we update distance for g? **No**



Suppose graph.edges returns edges in following order:

~~**(c, d)**~~, ~~**(a, b)**~~, ~~**(b, c)**~~, ~~**(g, f)**~~, ~~**(e, g)**~~, ~~**(g, e)**~~, ~~**(d, e)**~~, ~~**(e, f)**~~, ~~**(a, g)**~~

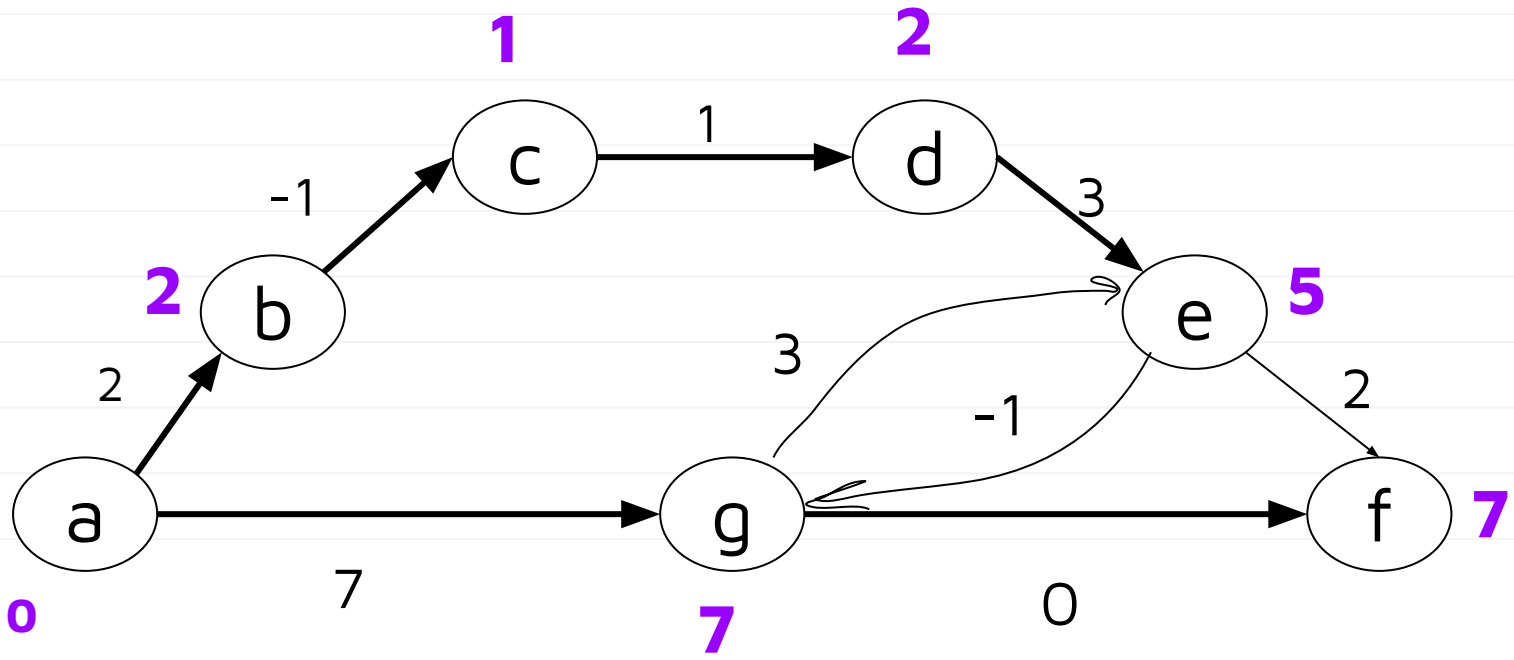
It was the end of the **second** iteration of BF



Suppose graph.edges returns edges in following order:

(c, d), (a, b), (b, c), (g, f), (e, g),
(g, e), (d, e), (e, f), (a, g)

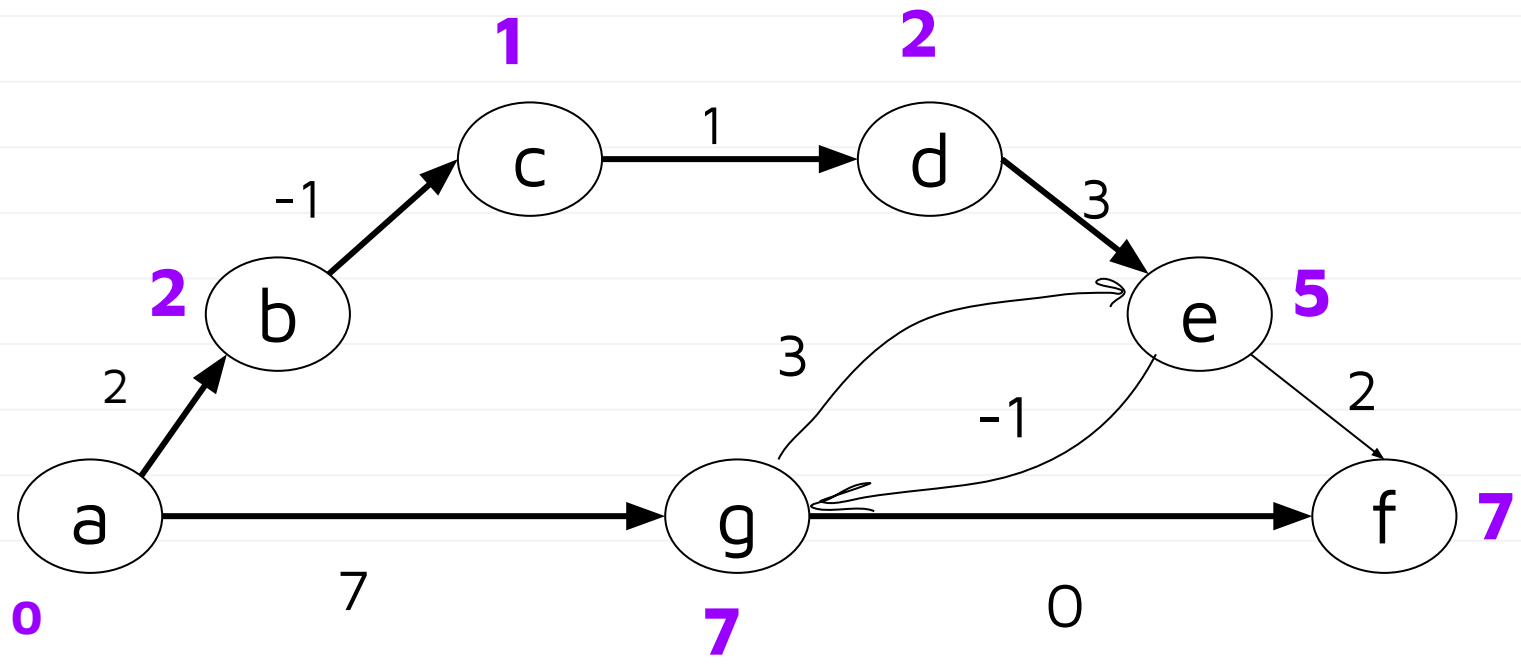
**One more time**



Suppose graph.edges returns edges in following order:

**(c, d)**, (a, b), (b, c), (g, f), (e, g), (g, e), (d, e), (e, f), (a, g)

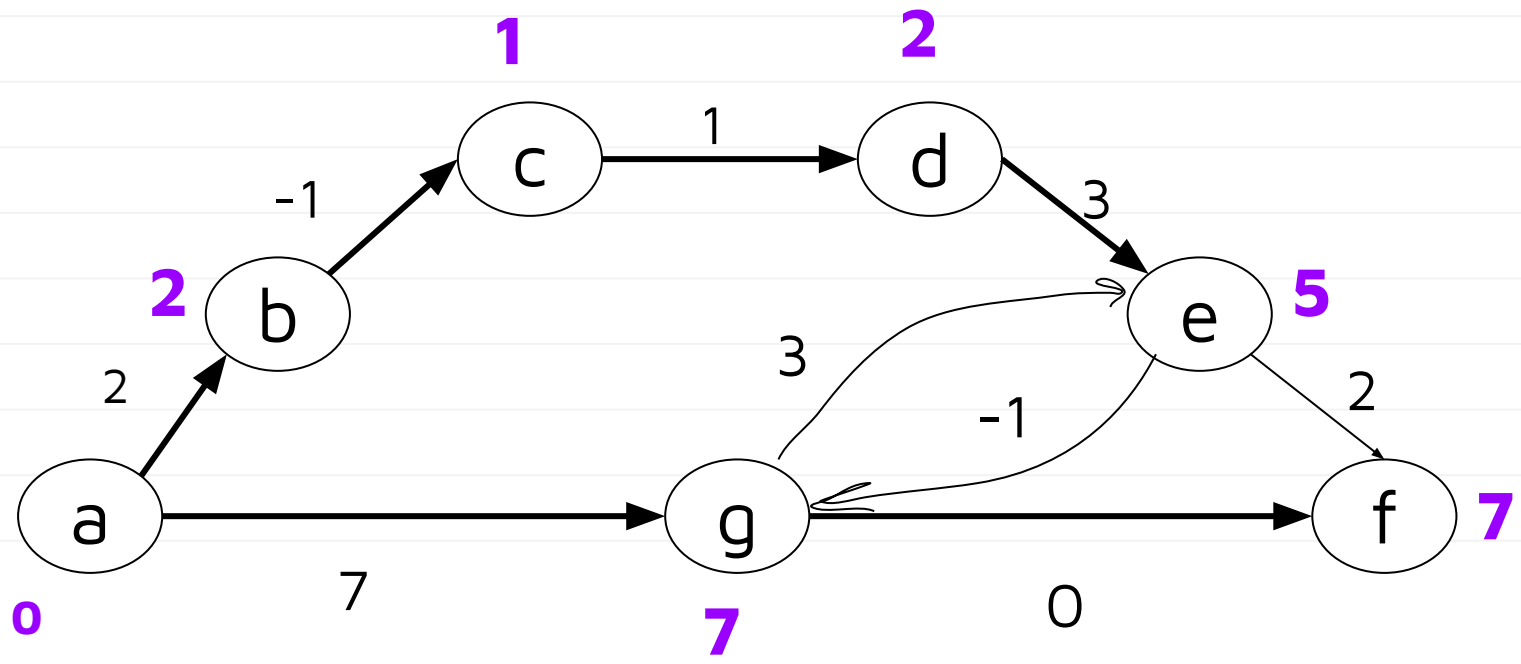
**No change**



Suppose graph.edges returns edges in following order:

**(c, d)**, **(a, b)**, (b, c), (g, f), (e, g), (g, e), (d, e), (e, f), (a, g)

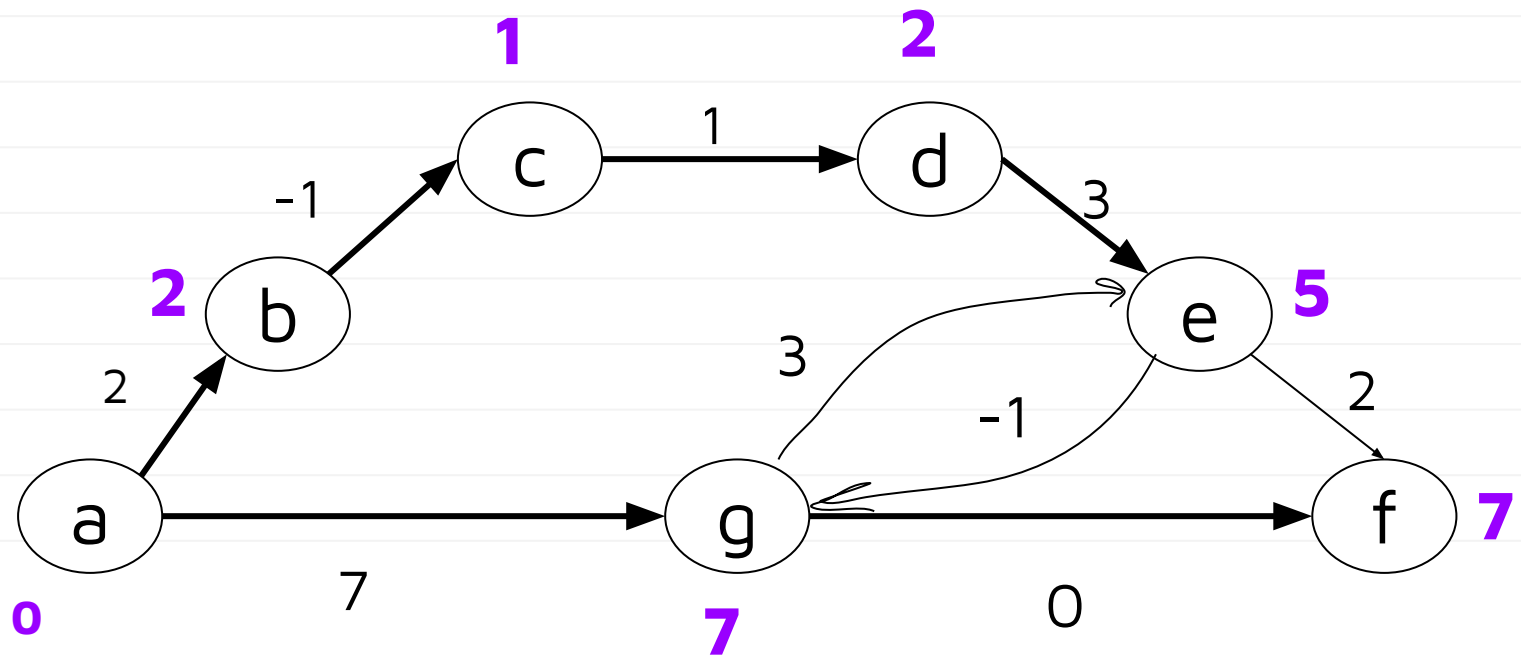
**No change**



Suppose graph.edges returns edges in following order:

**(c, d)**, **(a, b)**, **(b, c)**, (g, f), (e, g), (g, e), (d, e), (e, f), (a, g)

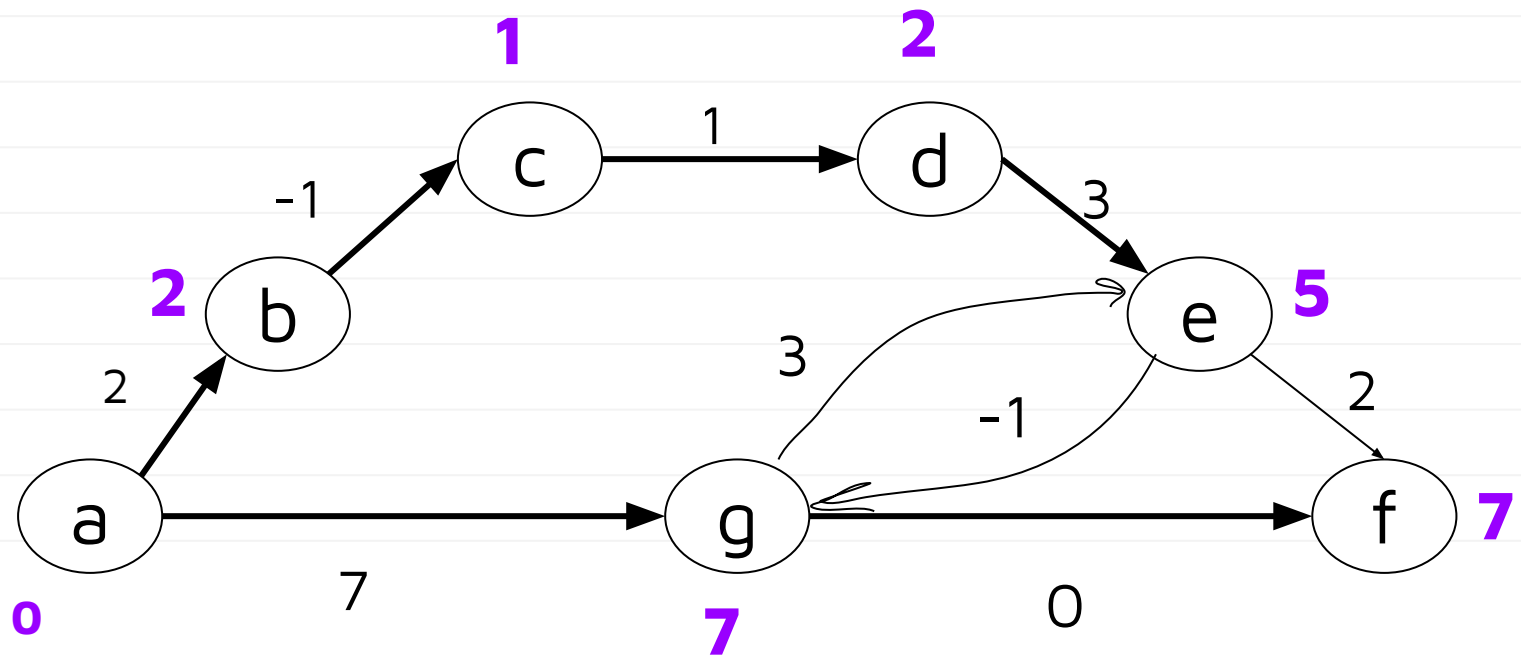
**No change**



Suppose graph.edges returns edges in following order:

**(c, d)**, **(a, b)**, **(b, c)**, **(g, f)**, (e, g), (g, e), (d, e), (e, f), (a, g)

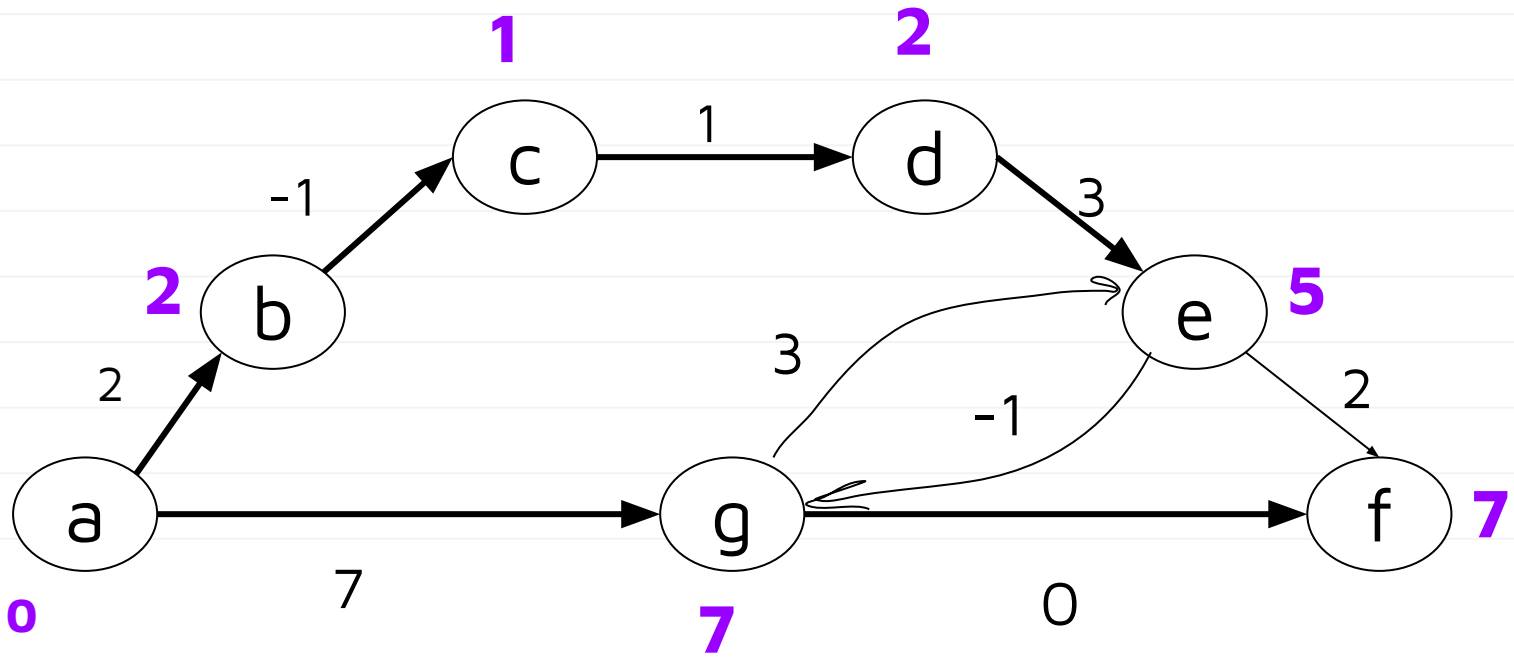
**No change**



Suppose graph.edges returns edges in following order:

**(c, d)**, **(a, b)**, **(b, c)**, **(g, f)**, **(e, g)**,
(g, e), (d, e), (e, f), (a, g)

**Update for node g.**



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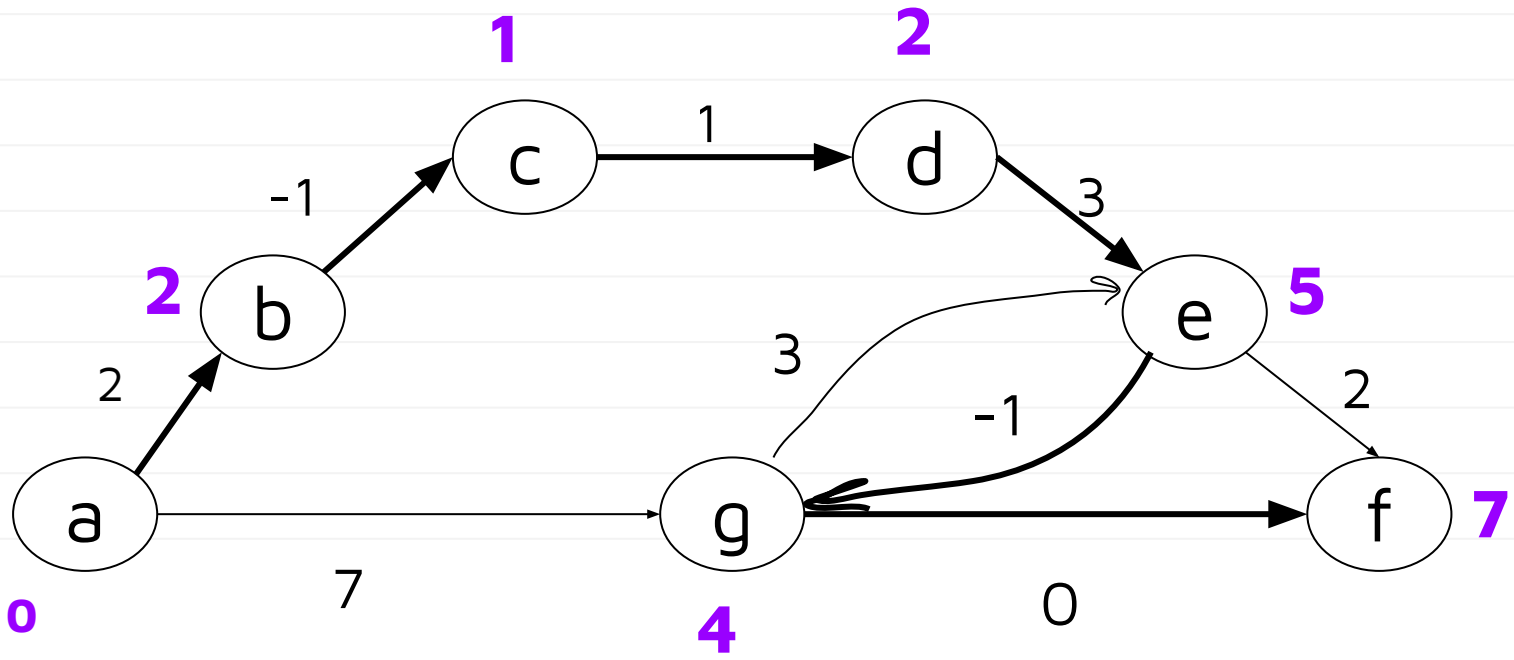
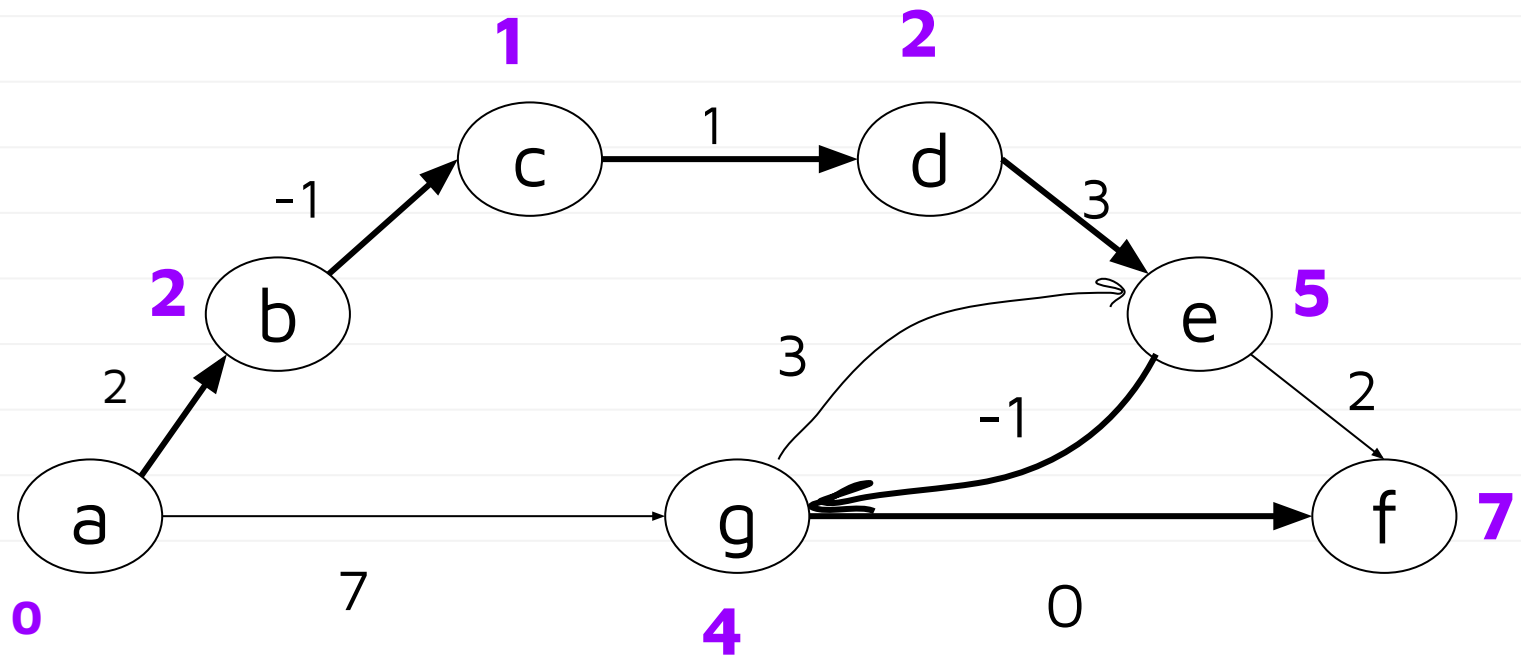


Suppose graph.edges returns edges in following order:

**(c, d)**, **(a, b)**, **(b, c)**, **(g, f)**, **(e, g)**, **(g, e)**, (d, e), (e, f), (a, g)

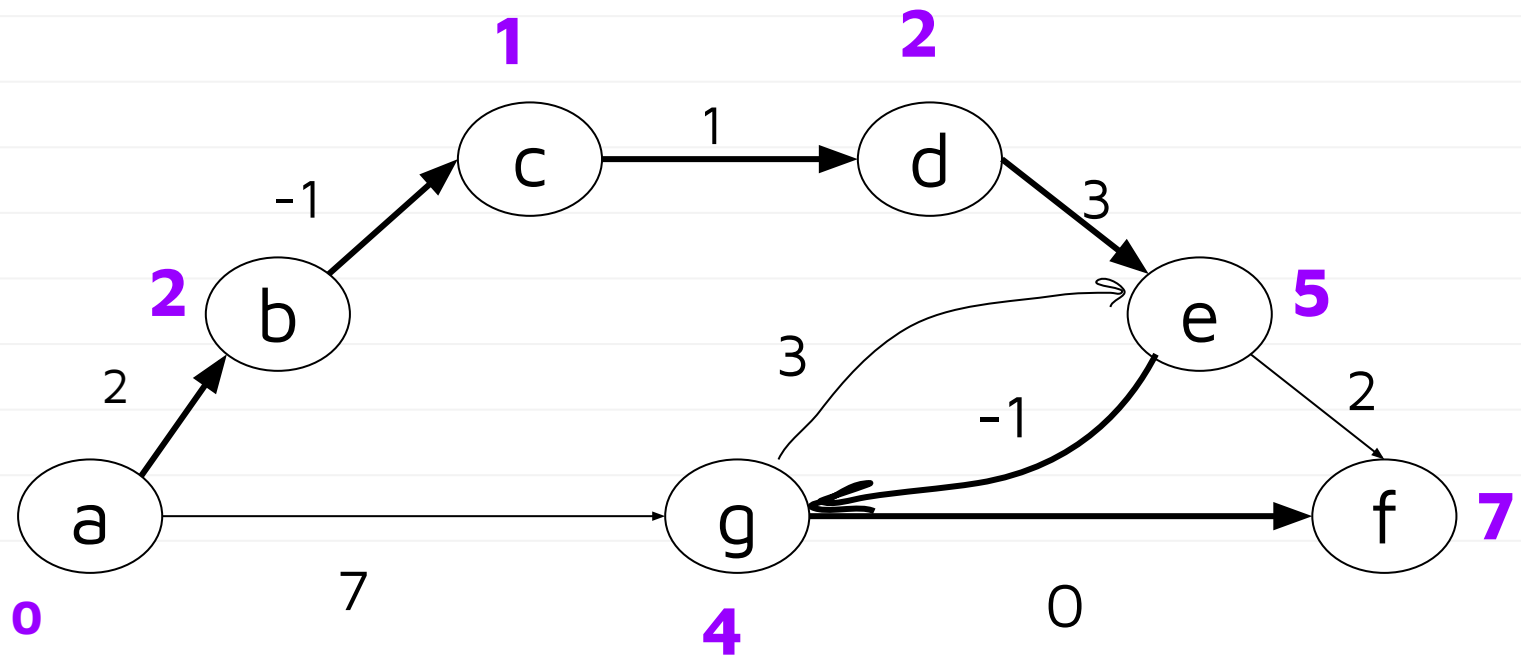
**No change**



Suppose graph.edges returns edges in following order:

**(c, d)**, **(a, b)**, **(b, c)**, **(g, f)**, **(e, g)**, **(g, e)**, **(d, e)**, (e, f), (a, g)

**No change**



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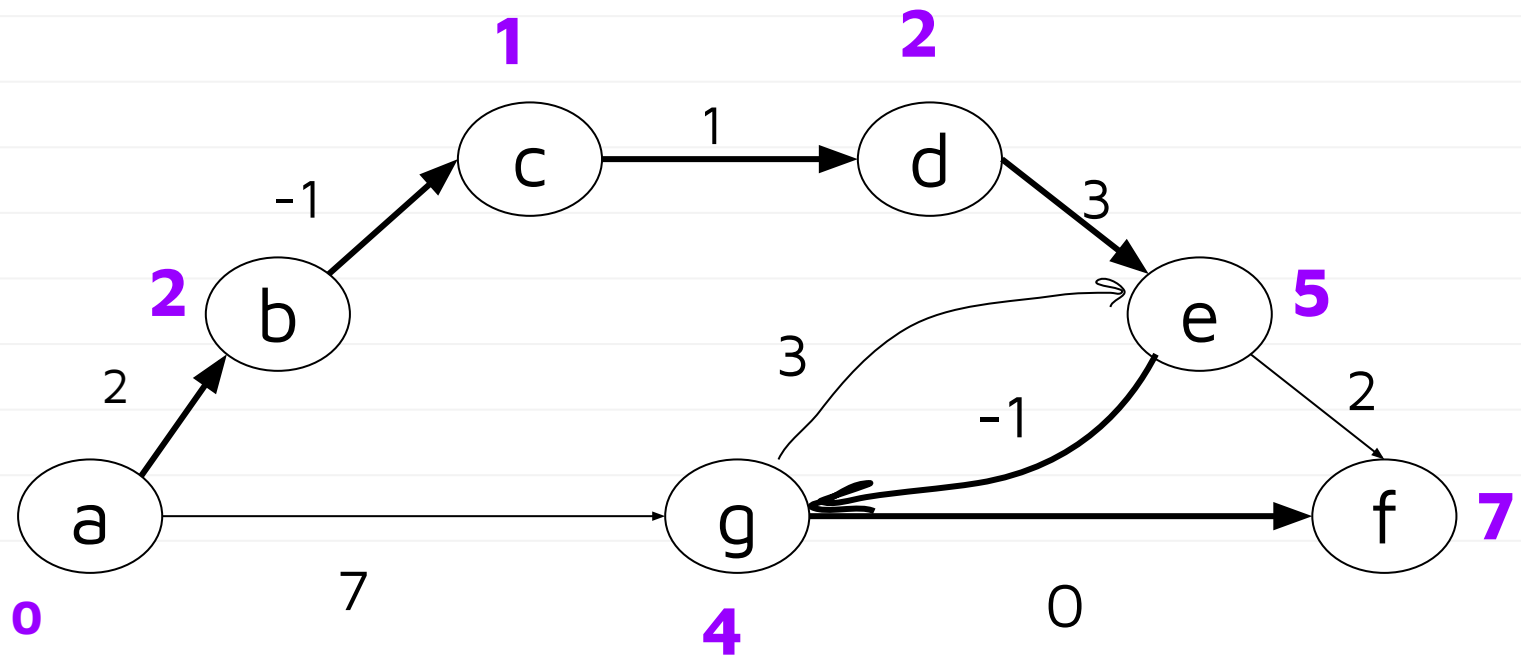
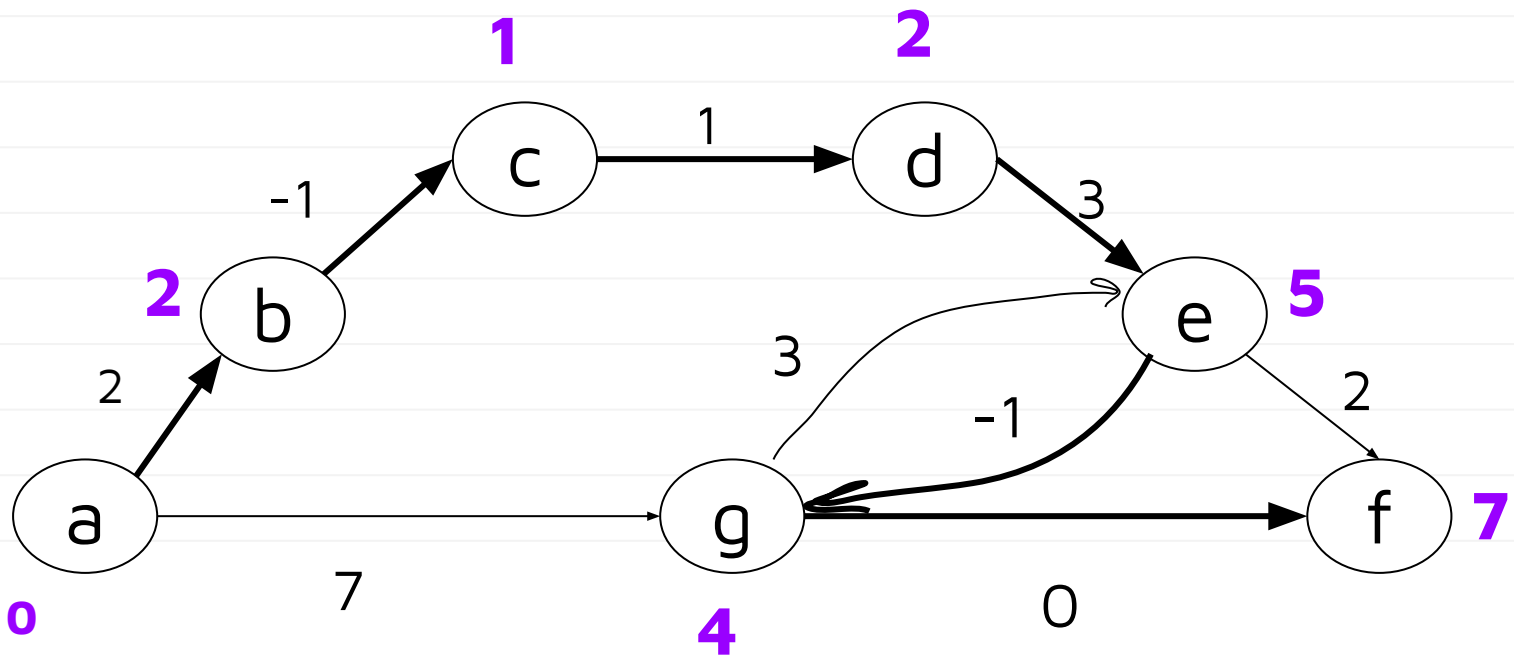


Suppose graph.edges returns edges in following order:

**(c, d), (a, b), (b, c), (g, f), (e, g), (g, e), (d, e), (e, f), (a, g)**

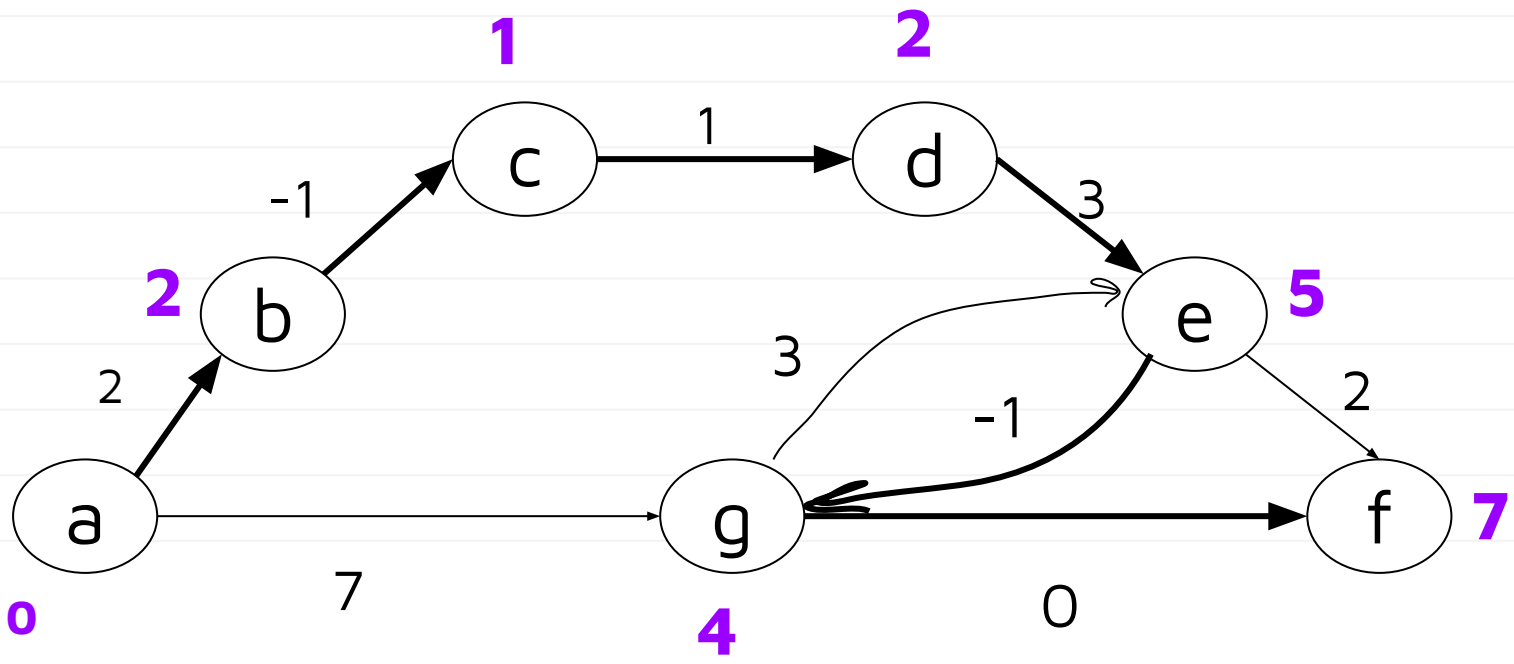
**No change**



Suppose graph.edges returns edges in following order:

**(c, d)**, **(a, b)**, **(b, c)**, **(g, f)**, **(e, g)**, **(g, e)**, **(d, e)**, **(e, f)**, **(a, g)**

It was the end of the **third** iteration of BF



Suppose graph.edges returns edges in following order:

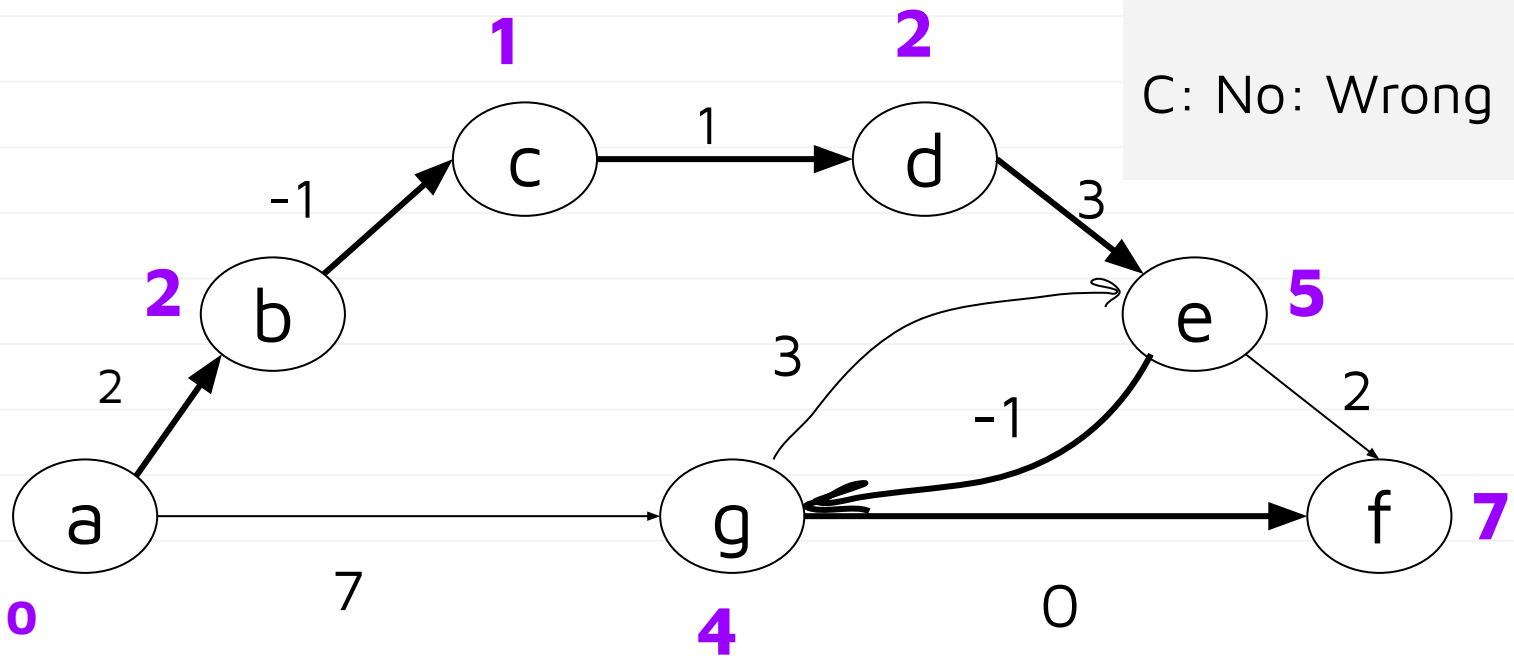
**(c, d), (a, b), (b, c), (g, f), (e, g), (g, e), (d, e), (e, f), (a, g)**

**Is everything look correct?**

A: Yes, we are done!

B: No: Wrong number

C: No: Wrong predecessor

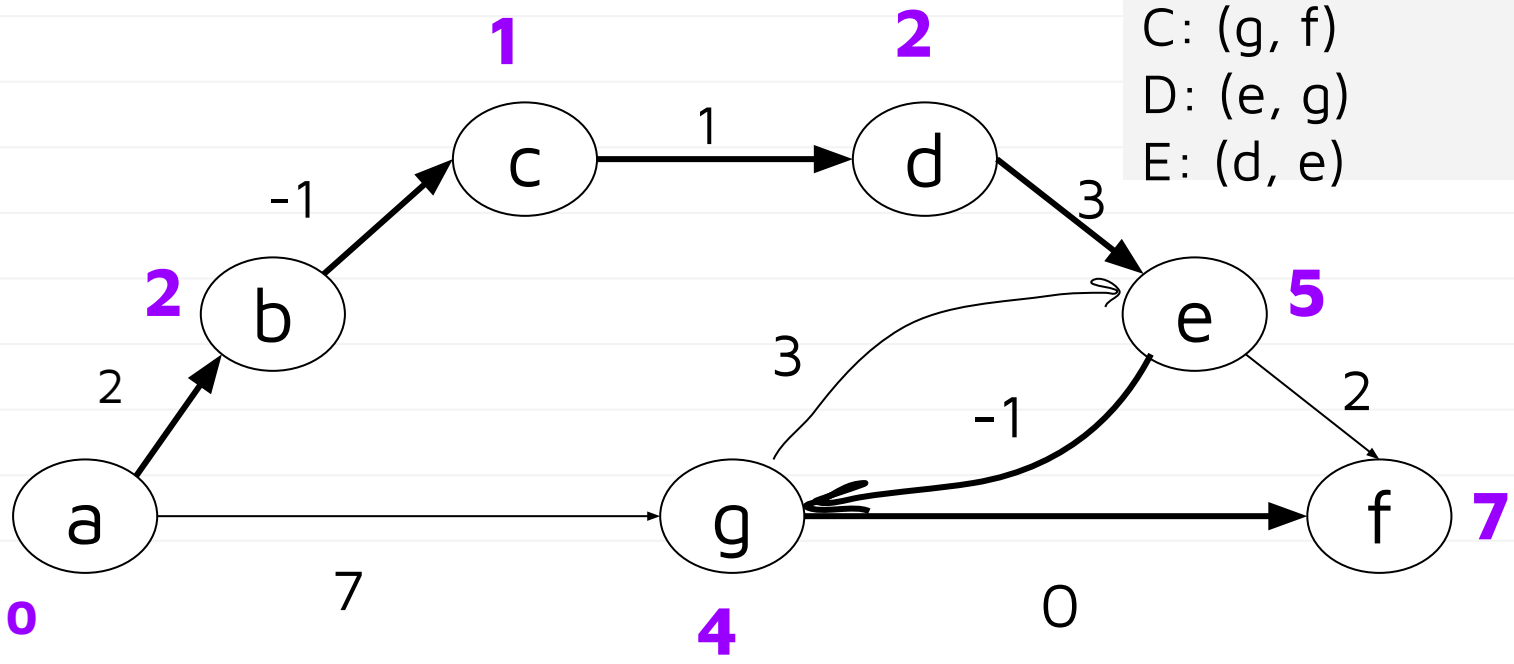


Suppose graph.edges returns edges in following order:

**(c, d), (a, b), (b, c), (g, f), (e, g), (g, e), (d, e), (e, f), (a, g)**

**What edge to we need to update?**

A: All of them
B: (e, f)
C: (g, f)
D: (e, g)
E: (d, e)



Suppose graph.edges returns edges in following order:

**(c, d), (a, b), (b, c), (g, f), (e, g), (g, e), (d, e), (e, f), (a, g)**

**What edge to we need to update?**

A: All of them
B: (e, f)
**C: (g, f)**
D: (e, g)
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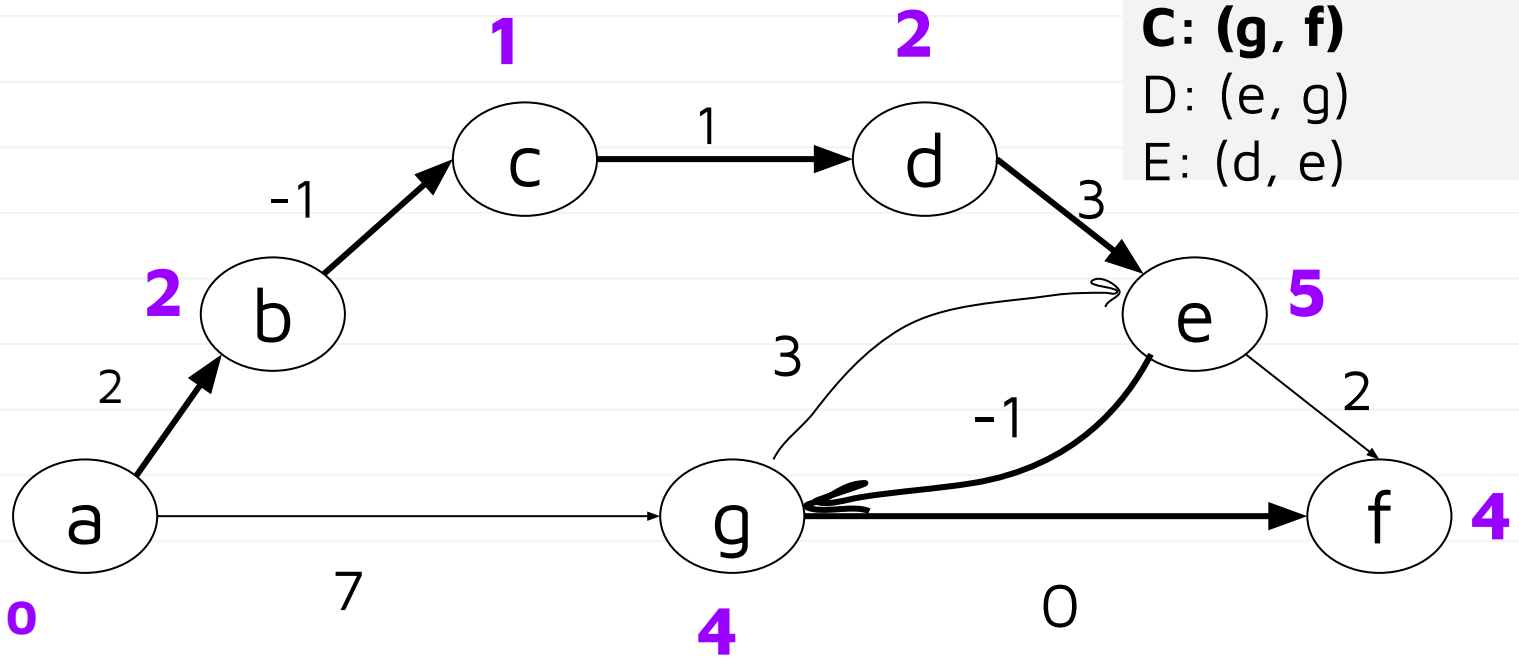
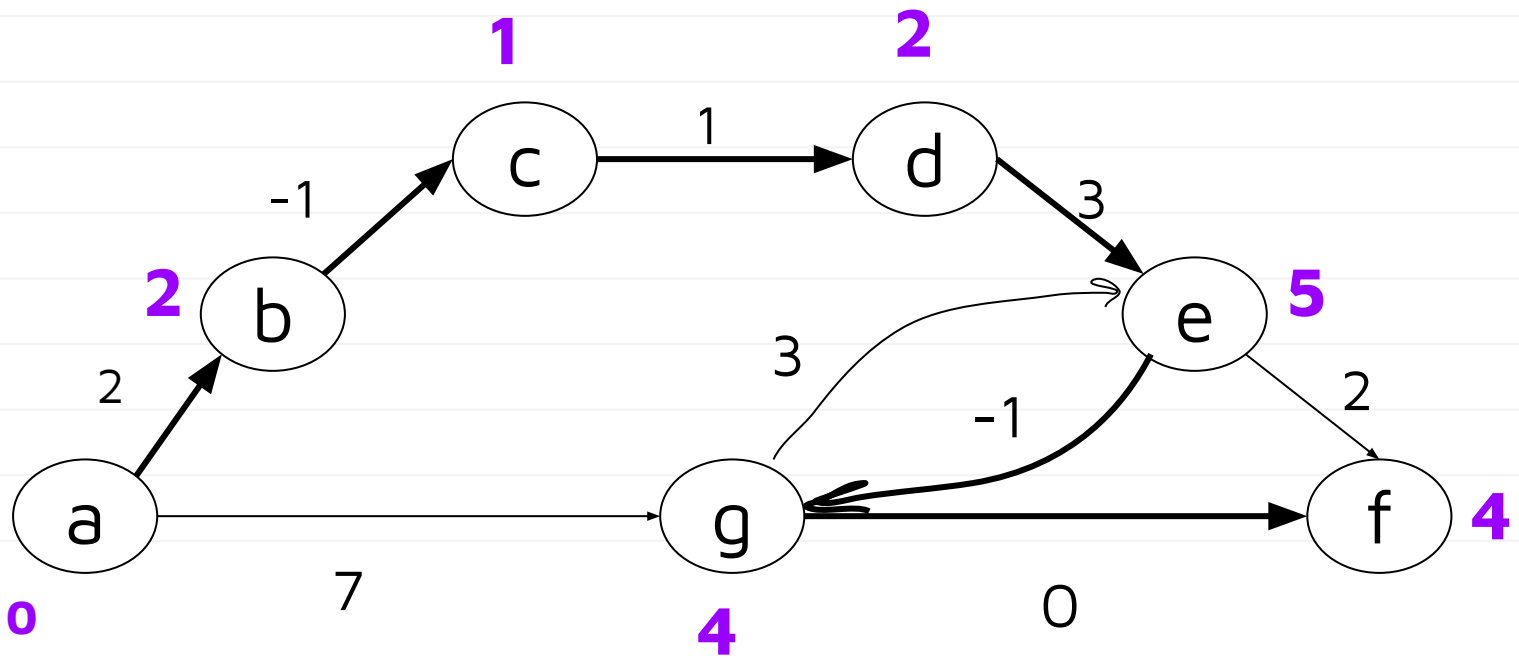


Suppose graph.edges returns edges in following order:

**(c, d), (a, b), (b, c), (g, f), (e, g), (g, e), (d, e), (e, f), (a, g)**

**How many total iterations did we perform?**

**4**

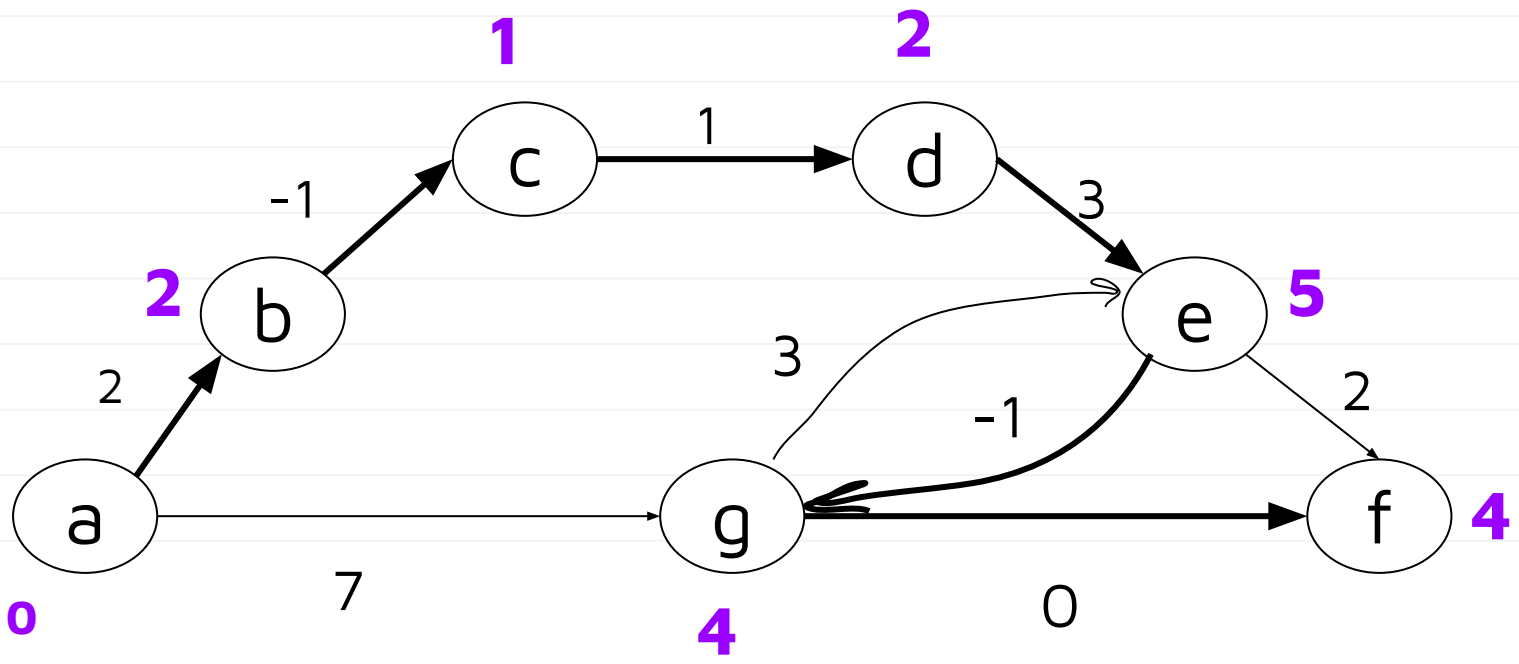


Suppose graph.edges returns edges in following order:

**(c, d), (a, b), (b, c), (g, f), (e, g), (g, e), (d, e), (e, f), (a, g)**

**Do we have to run one more iteration?**

**No**



# Loop Invariant

- **One** iteration: update *all* edges in arbitrary order.
- **Loop invariant**: After $\alpha$ iterations, all nodes whose ***actual*** shortest path from $s$ has $\leq \alpha$ edges are guaranteed to be estimated **correctly**.

# Bellman-Ford

- **Claim**: each node must have a shortest path which is simple (Edge case: cycles of weight zero)
- The most edges a simple path can have is $|V| - 1$
- **Idea of Bellman-Ford**: iteratively update all edges, repeat $|V| - 1$ times.

# The Bellman-Ford Algorithm

```
def bellman_ford(graph, weights, source):
"""Assume graph is directed."""
    est = {node: float('inf') for node in graph.nodes}
    est[source] = 0
    predecessor = {node: None for node in graph.nodes}

    for i in range(len(graph.nodes)-1):
        for (u, v) in graph.edges:
            update(u, v, weights, est, predecessor)
    return est, predecessor
```

# The Bellman-Ford Algorithm: Complexity

```
def bellman_ford(graph, weights, source):
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```

Setup: _____time:

Each update takes _____time

There are exactly _____updates

Total time complexity:_____

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```

Setup: $\Theta(V)$ time:

Each update takes _____ time

There are exactly _____ updates

Total time complexity: _____

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Setup: $\Theta(V)$ time

Each update takes $\Theta(1)$ time

There are exactly _____ updates

Total time complexity: _____

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Setup: $\Theta(V)$ time

Each update takes $\Theta(1)$ time

There are exactly $(|V|-1) \times |E|$ updates

Total time complexity: _____

# The Bellman-Ford Algorithm: Complexity

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```

Setup: $\Theta(V)$ time

Each update takes $\Theta(1)$ time

There are exactly $(|V|-1) \times |E|$ updates

Total time complexity: $\Theta(V + VE)$

# Early Stopping and Negative Cycles

# Early Stopping

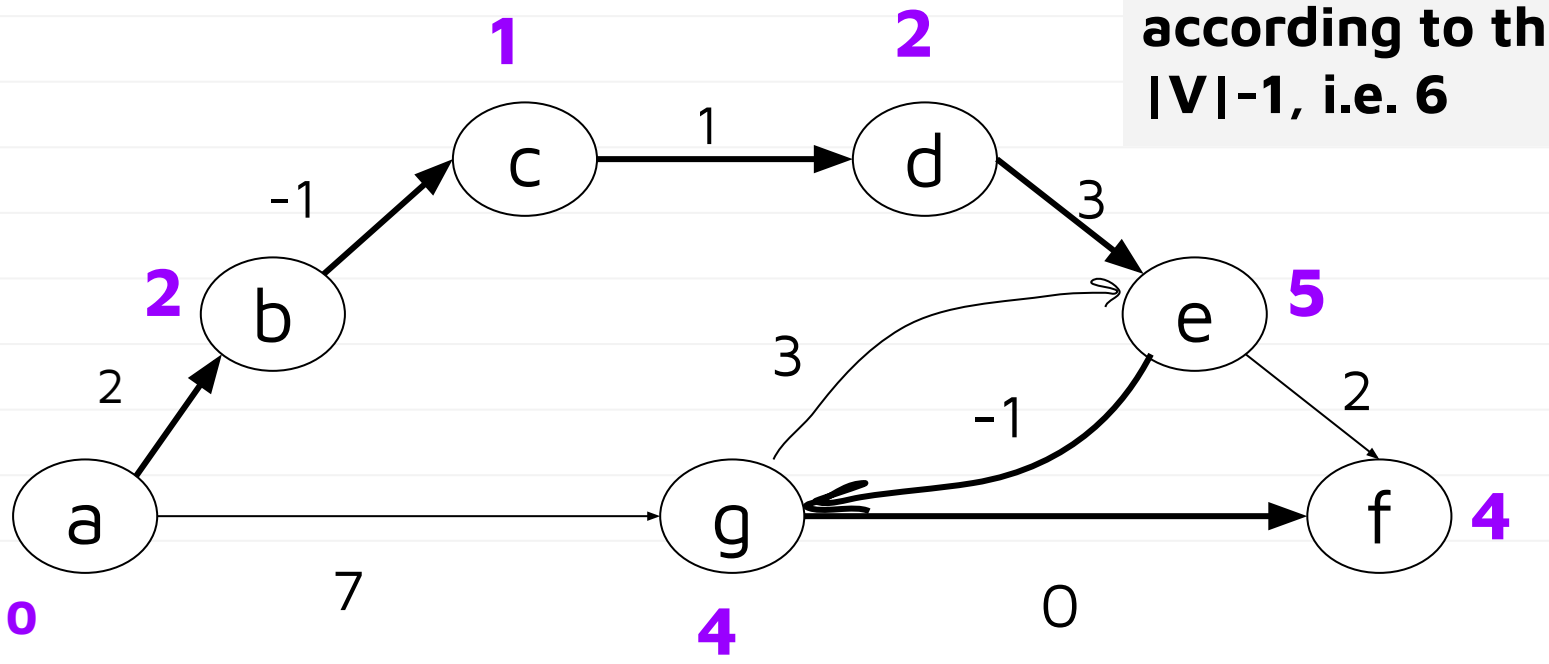
- B-F may not need to run for $|V| - 1$ iterations.
- If no predecessors change, we can break:

Suppose graph.edges returns edges in following order:

**(c, d), (a, b), (b, c), (g, f), (e, g), (g, e), (d, e), (e, f), (a, g)**

**How many total iterations did we perform?**
**4**

**How many iterations according to the code?**
**|V|-1, i.e. 6**



# Early Stopping

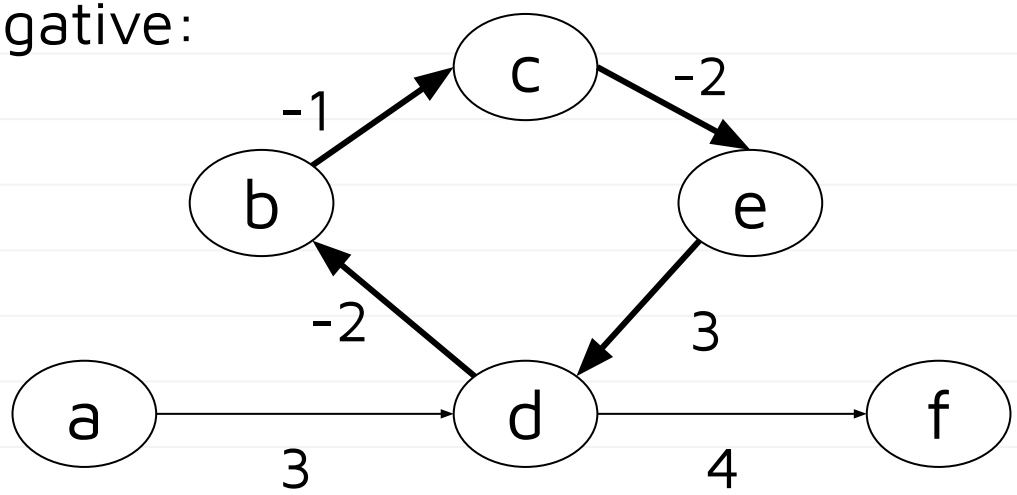
- B-F may not need to run for $|V| - 1$ iterations.
- If no predecessors change, we can break:

```
def bellman_ford(graph, weights, source):
    """Early stopping version."""
    est = {node: float('inf') for node in graph.nodes}
    est[source] = 0
    predecessor = {node: None for node in graph.nodes}

    for i in range(len(graph.nodes) - 1):
        any_changes = False
        for (u, v) in graph.edges:
            changed = update(u, v, weights, est, predecessor)
            any_changes = changed or any_changes
        if not any_changes:
            break
    return est, predecessor
```

# Negative Cycles

- A **negative cycle** is a cycle whose total edge weight is negative:



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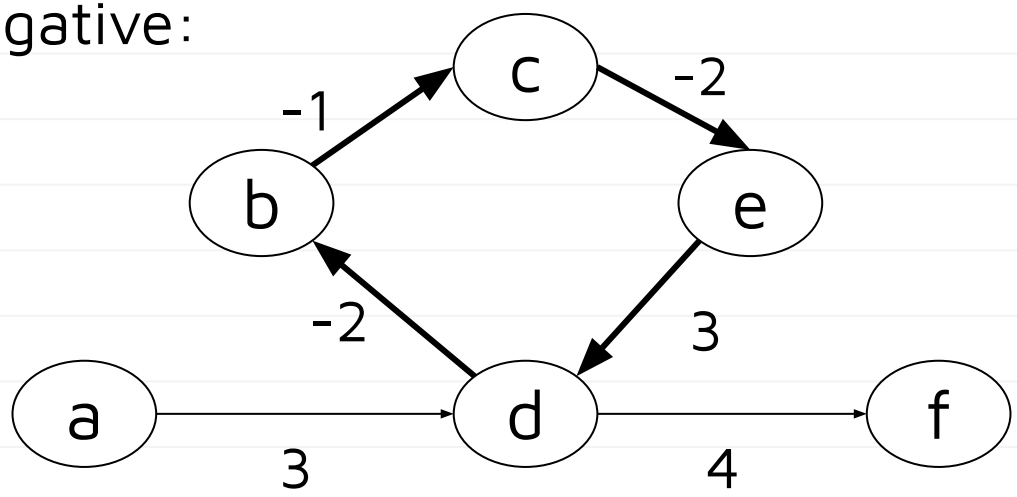


- If a graph has a negative cycle, (some) shortest paths are **not well defined.**

# Detecting Negative Cycles

- If graph **does not have** negative cycles, estimated distances eventually stop changing (after at most $|V| - 1$ iterations).
- If graph **has** negative cycles, estimated distances *always* decrease.
- To detect them: run a $|V|$th iteration; if distances change, a negative cycle exists.

# Detecting Negative Cycles

```python
def bellman_ford(graph, weights, source):
    """Early stopping version, detects negative cycles."""
    est = {node: float('inf') for node in graph.nodes}
    est[source] = 0
    predecessor = {node: None for node in graph.nodes}

    for i in range(len(graph.nodes)):
        any_changes = False
        for (u, v) in graph.edges:
            changed = update(u, v, weights, est, predecessor)
            any_changes = changed or any_changes
        if not any_changes:
            break
    # this will be True if negative cycles exist
    contains_negative_cycles = any_changes
    return est, predecessor, contains_negative_cycles
```

# Thank you!

**Do you have any questions?**

CampusWire!