DSC 40B

Lecture 13 : Binary Search Trees

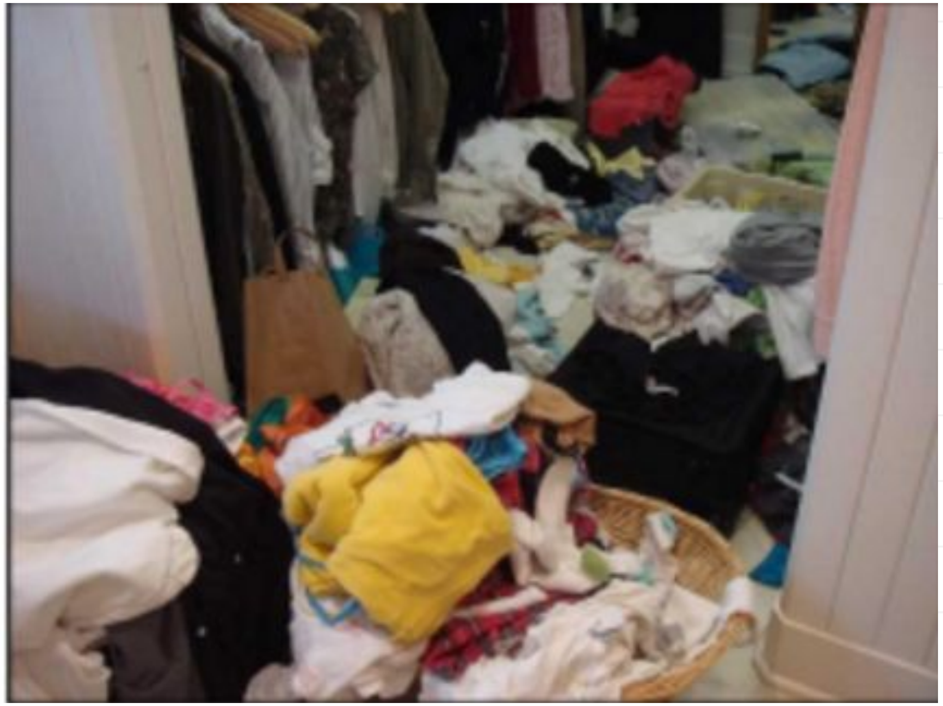
Dynamic Sets

Clothes

- How do you store your clothes?

Clothes

- How do you store your clothes?



Clothes

- How do you store your clothes?

Clothes: Tradeoffs

- **Messy**:
  - No upfront cost.
  - Cost to search is high.
- **Organized**
  - Big upfront cost.
  - Cost to search is low.
- "Right" choice depends on how **often** we search.

Data Structures and Algorithms

- **Data structures** are ways of organizing data to make certain operations *faster*.
- Come with an upfront cost (**preprocessing**).
- "Right" choice of data structure depends on what operations we'll be doing in the future.

Queries: Easy to Hard

- We've been thinking about **queries**.
  - Given a collection of data, is x in the collection?
- Querying is a fundamental operation.
  - Useful in a data science sense.
  - But also frequently performed in algorithms.
- There are *several* situations to think about.

Situation #1: Static Set, One Query

- **Given**: an **unsorted** collection of n numbers (or strings, etc.).
- In future, you will be asked **single** query.
- Which is better: *linear search* or *sort + binary search*?

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- Which is better: *linear search* or *sort + binary search*?

A: *linear search*

B: *sort + binary search*

C: both are equivalent

Situation #1: Static Set, One Query

- **Given**: an **unsorted** collection of n numbers (or strings, etc.).
- In future, you will be asked **single** query.
- Which is better: *linear search* or *sort + binary search*?
- Linear search: $\Theta(n)$ worst case.
- Binary search would require sorting first in $\Theta(n \log n)$ worst case

Situation #2: Static Set, Many Queries

- **Given**: an unsorted collection of n numbers (or strings, etc.).
- In future, you will be asked **many** queries.
- Which is better: `linear search` or `sort + binary search`?

A: *linear search*

B: *sort + binary search*

C: both are equivalent

D: Depends on the number of queries

Situation #2: Static Set, Many Queries

- **Given**: an unsorted collection of n numbers (or strings, etc.).
- In future, you will be asked **many** queries.
- Which is better: `linear search` or `sort + binary search`?
- Depends on number of queries!

Exercise

- Suppose you have a static set of n items. How long will it take* a to perform k queries in total with:
  - linear search?
  - sort + binary search?

- If $k = n/10$, which should you use? What if $k = \log n$?

**On average. Assume the best case is rare.*

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$T_{linear}(n) = kn$ # *to perform k searches using linear search.*

Exercise

- Suppose you have a static set of n items. How long will it take* a to perform k queries in total with:
  - linear search?
  - sort + binary search?

$T_{liner}(n) = kn$ # *to perform k searches using linear search.*

$T_{binaryS}(n) = n \log n + k \log n$ # *to perform k searches using binary search.*

Exercise, k = n/10

- Suppose you have a static set of n items. How long will it take* a to perform k queries in total with:
  - linear search?
  - sort + binary search?

$T_{liner}(n)$ = kn # *to perform k searches using linear search.*

$T_{binaryS}(n)$ = n logn + k log n # *to perform k searches using binary search.*

A: *linear search*

B: *sort + binary search*

C: both are equivalent

Exercise, k = n/10

- Suppose you have a static set of n items. How long will it take* a to perform k queries in total with:
  - linear search?
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$T_{liner}(n)$ = kn # *to perform k searches using linear search.*

$T_{binaryS}(n)$ = n logn + k log n # *to perform k searches using binary search.*

$T_{liner}(n)$ = $\mathbf{n^2/10}$ vs $T_{binaryS}(n)$ = n logn + **n /10 * logn**

Exercise, $k = \log n$

- Suppose you have a static set of n items. How long will it take* a to perform k queries in total with:
    - linear search?
    - sort + binary search?

$T_{liner}(n)$ = kn # *to perform k searches using linear search.*

$T_{binaryS}(n)$ = n logn + k log n # *to perform k searches using binary search.*

$T_{liner}(n)$ = **n logn**   vs $T_{binaryS}(n)$ = **n logn** + logn * logn

Situation #3: Dynamic Set, Many Queries

- **Given**: a collection of n numbers (or strings, etc.).
- In future, you will be asked **many** queries and to insert new elements.
- Best approach: ?

Situation #3: Dynamic Set, Many Queries

- **Given**: a collection of n numbers (or strings, etc.).
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- Best approach: ?

Binary Search?

- Can we still use binary search?
- **Problem**: To use binary search, we **must** maintain array in sorted order as we insert new elements.
- Inserting into array takes $\Theta(n)$ time in worst case.
    - Must "make room" for new element.
    - Can we use linked list with binary search?

Midterm: mic

- Up to (including) this lecture
- This classroom
- Same time as a the class time.

Last Lecture:

- **Given**: a collection of n numbers (or strings, etc.).
- **Dynamic**: we can add/remove elements
- **Queries**:, you will be asked **many** queries
- Best approach: ?

Last Lecture:

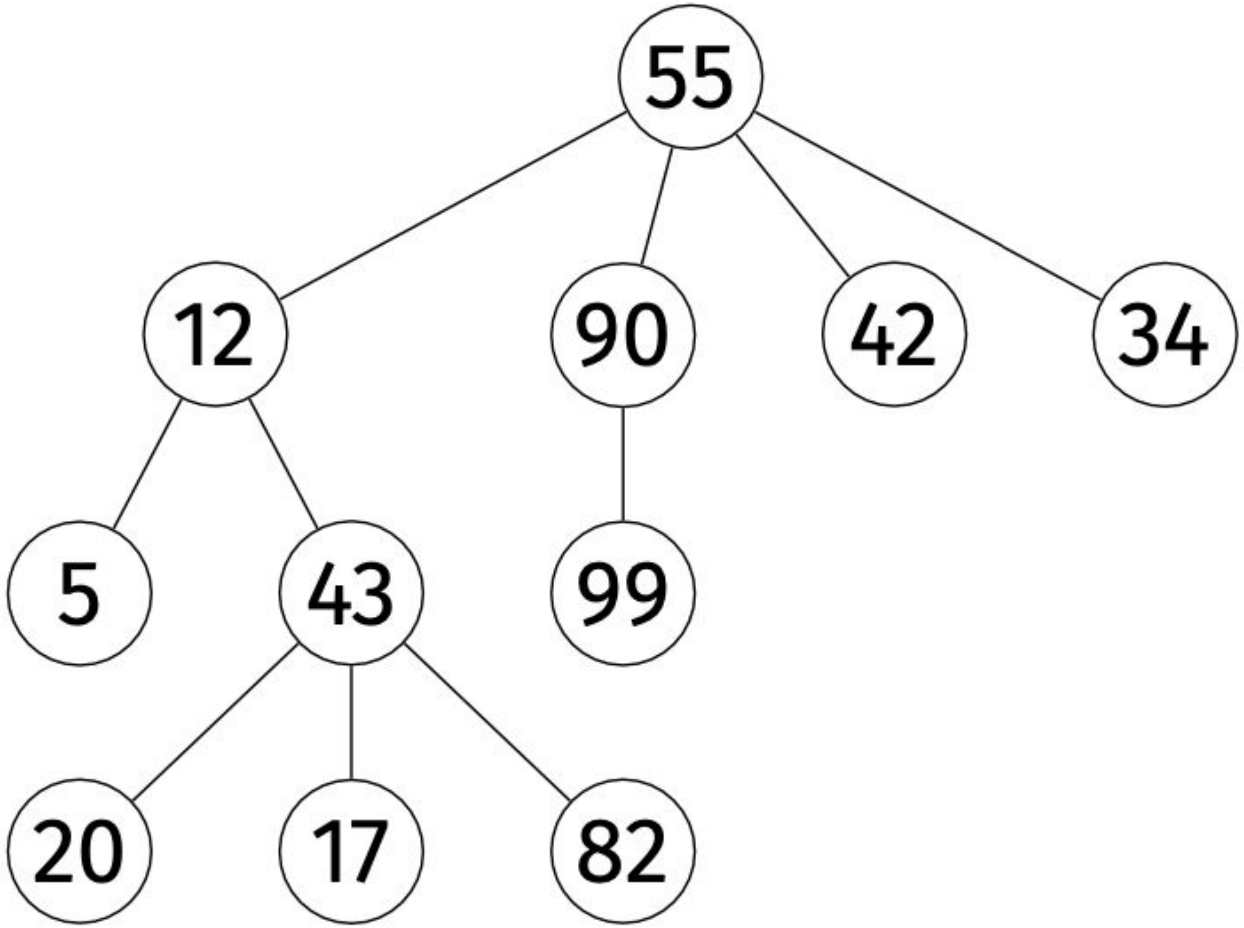
- **Given**: a collection of n numbers (or strings, etc.).
- **Dynamic**: we can add/remove elements
- **Queries**:, you will be asked **many** queries
- **Approaches**:
  - Linear Search -> not efficient with many searches.
  - Sort + binary search -> the set is changing, need to re-sort every time.

Today:

- Introduce (or review) **binary search trees.**
- BSTs support **fast** *queries* and *insertions and deletions*.
- **Preserve** sorted order of data after insertion.
- Can be modified to solve **many** problems efficiently.
    - Example: finding order statistics.

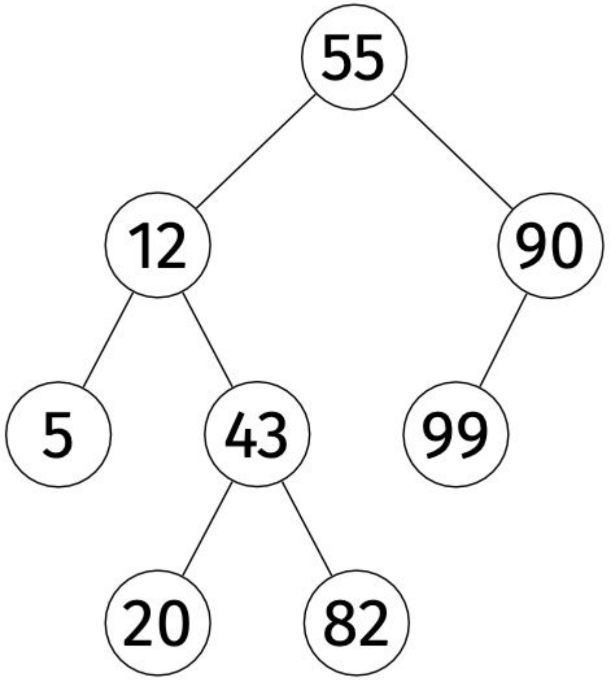
Binary Search Trees

Trees



Binary Trees

Each node has **at most** two children (left and right).



Binary Search Tree

- A **binary search tree** (BST) is a ***binary*** tree that satisfies the following for **any** node x:
- if y is in x's **left** subtree:

  y.key ≤ x.key

- if y is in x's **right** subtree:

  y.key ≥ x.key

Assumption (for simplicity)

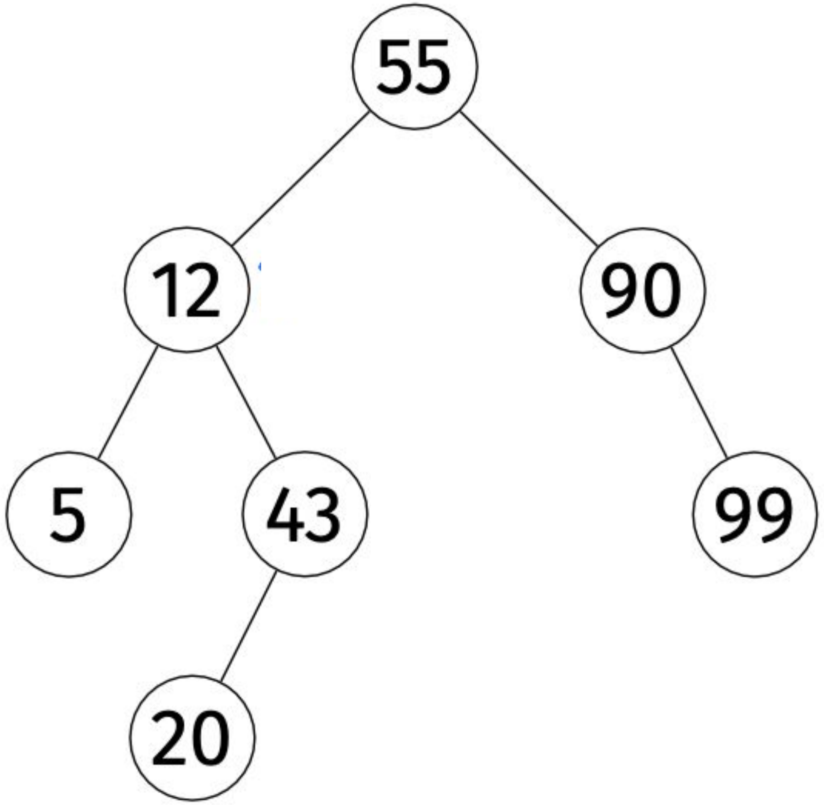
- We'll assume keys are **unique** (no duplicates).
- if y is in x's **left** subtree:

  y.key < x.key

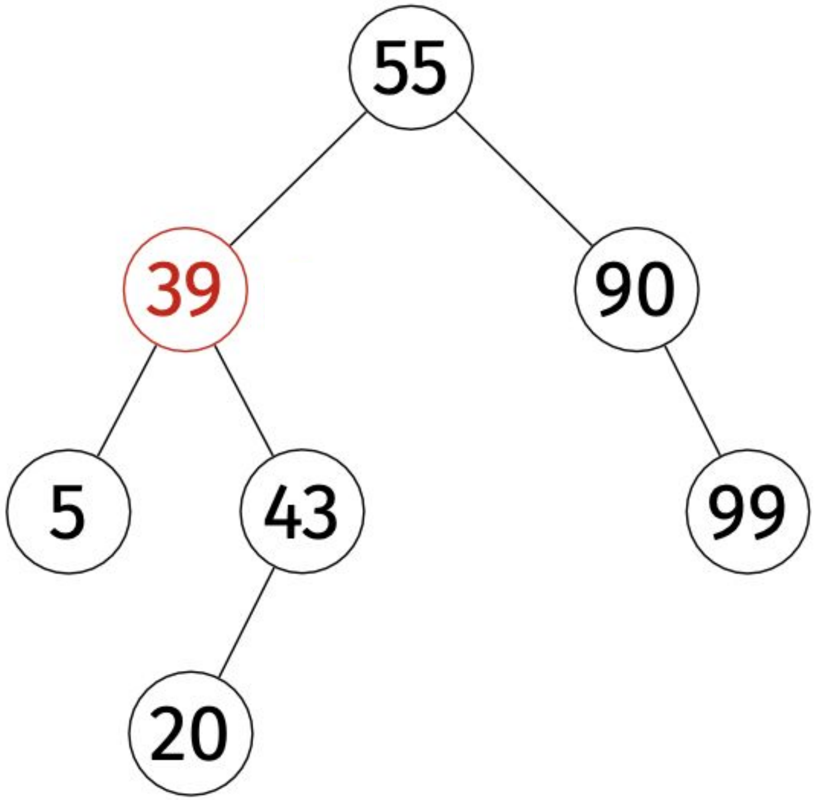
- if y is in x's **right** subtree:

  y.key > x.key

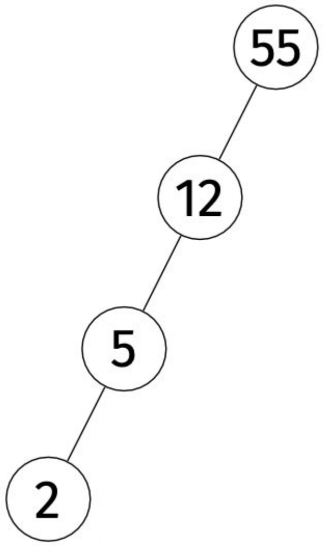
Example: This is a BST.



Example: not a BST



Think: Is this is a BST?



A: Yes

B: No

C: Maybe

D: Not paying attention

Height

- The **height** of a tree is the number of edges on the **longest** path from the root to a leaf.
- Suppose a binary tree has n nodes.
- The **tallest** it can be is $\approx n$
- The **shortest** it can be is $\approx$?

A: 1

B: log n

C: n

D: n log n

Height

- The **height** of a tree is the number of edges on the **longest** path from the root to a leaf.
- Suppose a binary tree has n nodes.
- The **tallest** it can be is $\approx n$
- The **shortest** it can be is $\approx \log_2 n$

In Python

```
class Node:
    def __init__(self, key, parent=None):
        self.key = key
        self.parent = parent
        self.left = None
        self.right = None

class BinarySearchTree:
    def __init__(self, root: Node):
        self.root = root
```

In Python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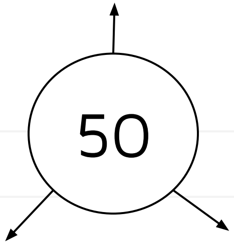


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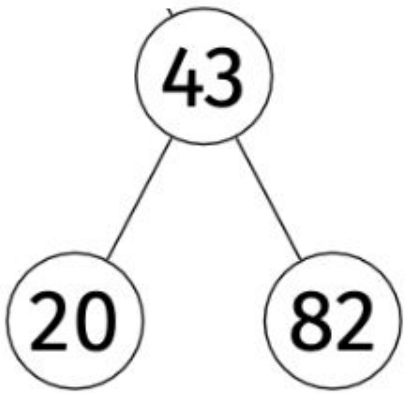



```
root = Node(43)
n1 = Node(20 , parent=root)
root.left = n1
n2 = Node(82 , parent=root)
root.right = n2
tree = BinarySearchTree(root)
```

Queries and Insertions in BSTs

Why?

- BSTs impose structure on data.
- “Not quite sorted”.
- Preprocessing for making insertions and queries faster.

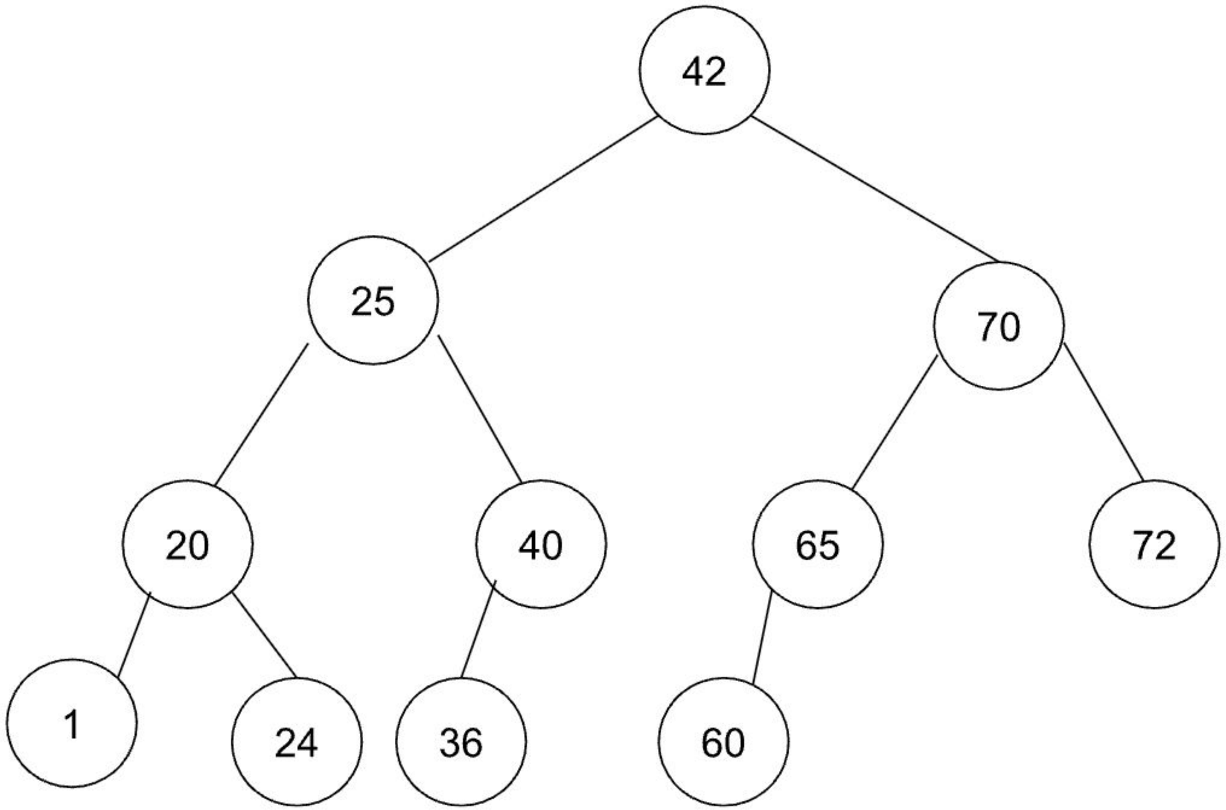
Operations on BSTs

- We will want to:
  - **query** a key (is it in the tree?)
  - **insert** a new key

Queries

- **Given**: a BST and a target, t.
- **Return**: *True* or *False*, is the target in the collection?

Queries: 36, 65, 23?



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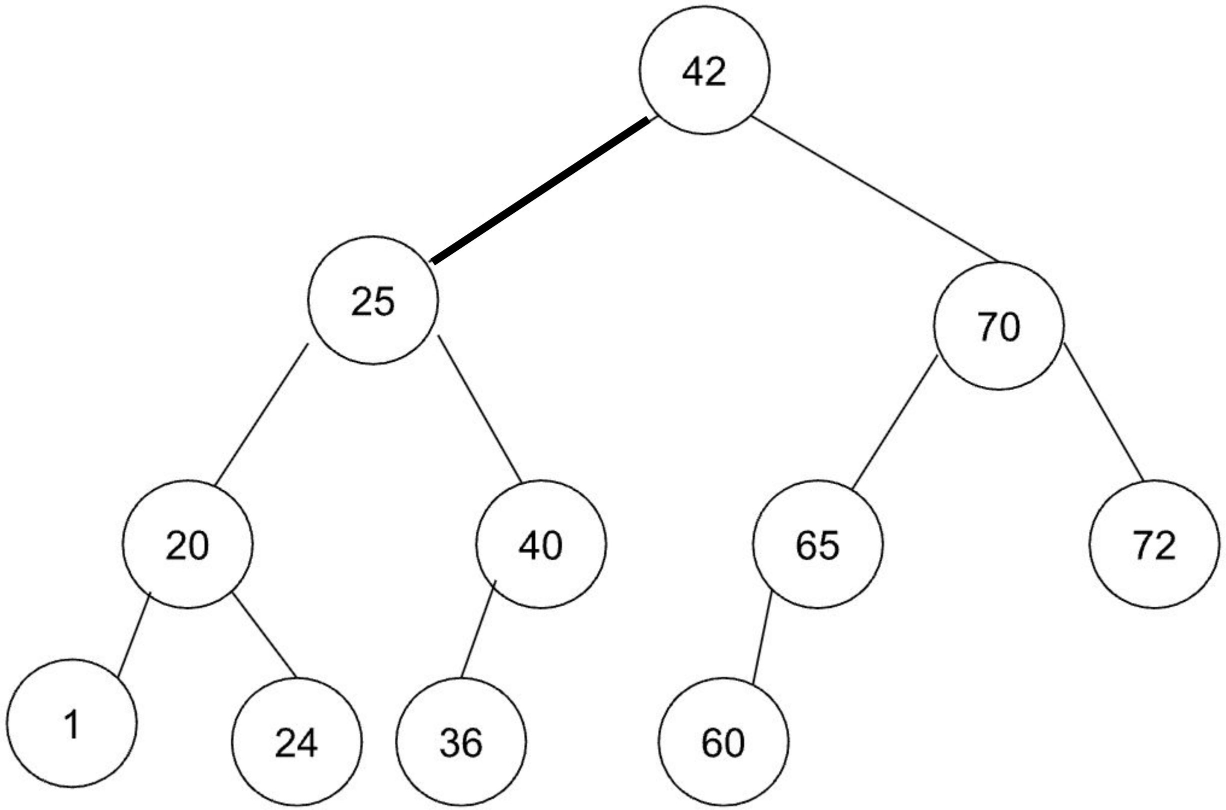

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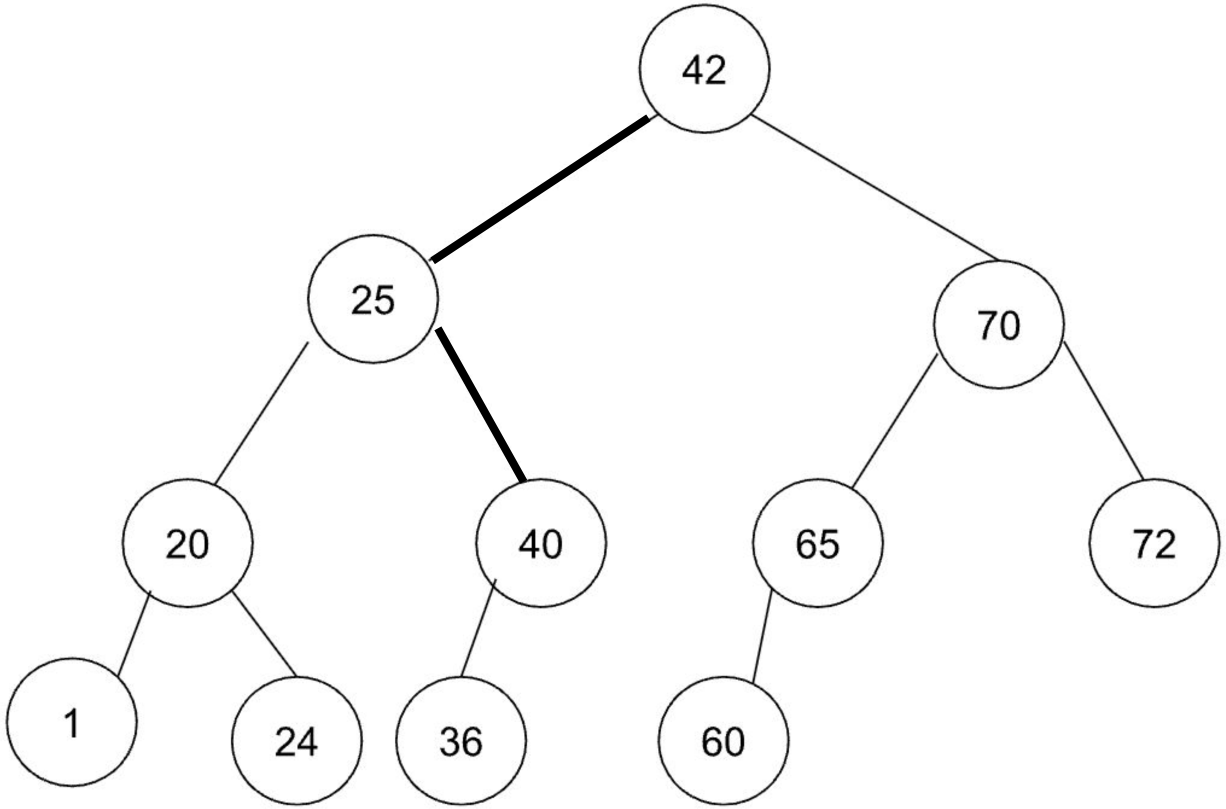

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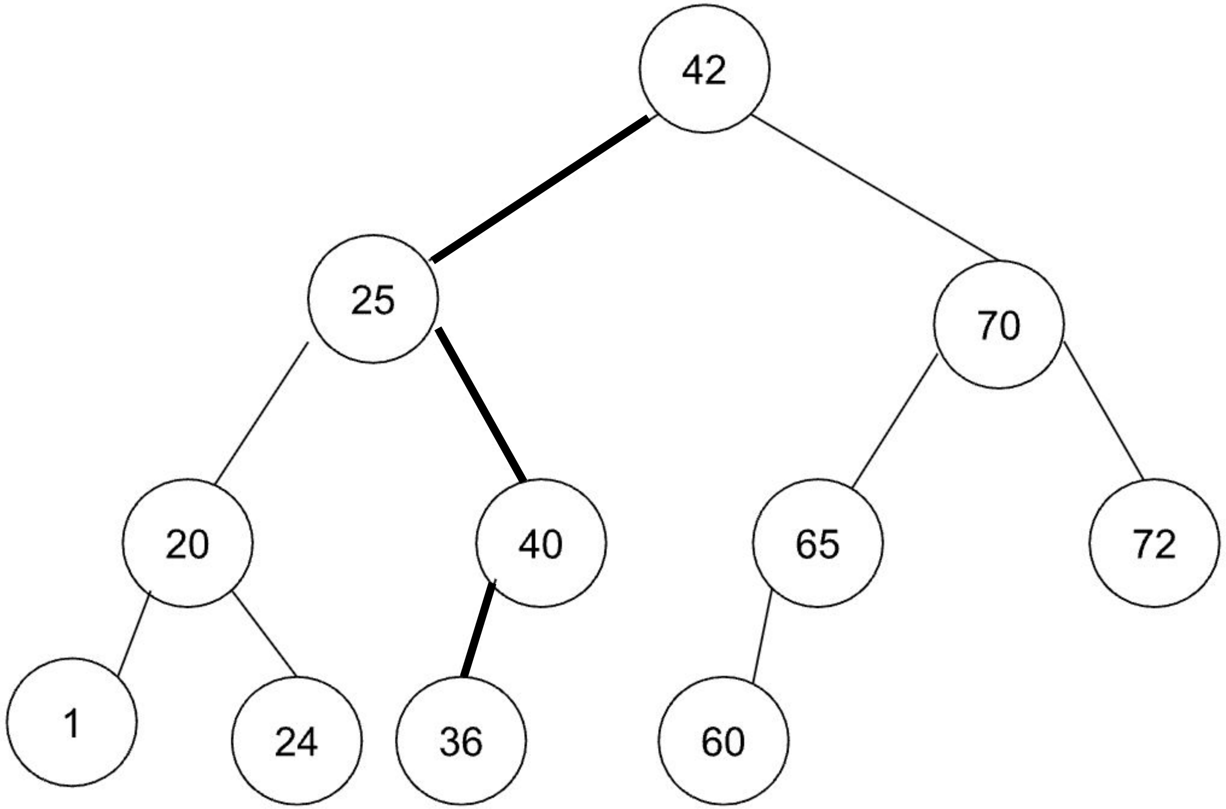

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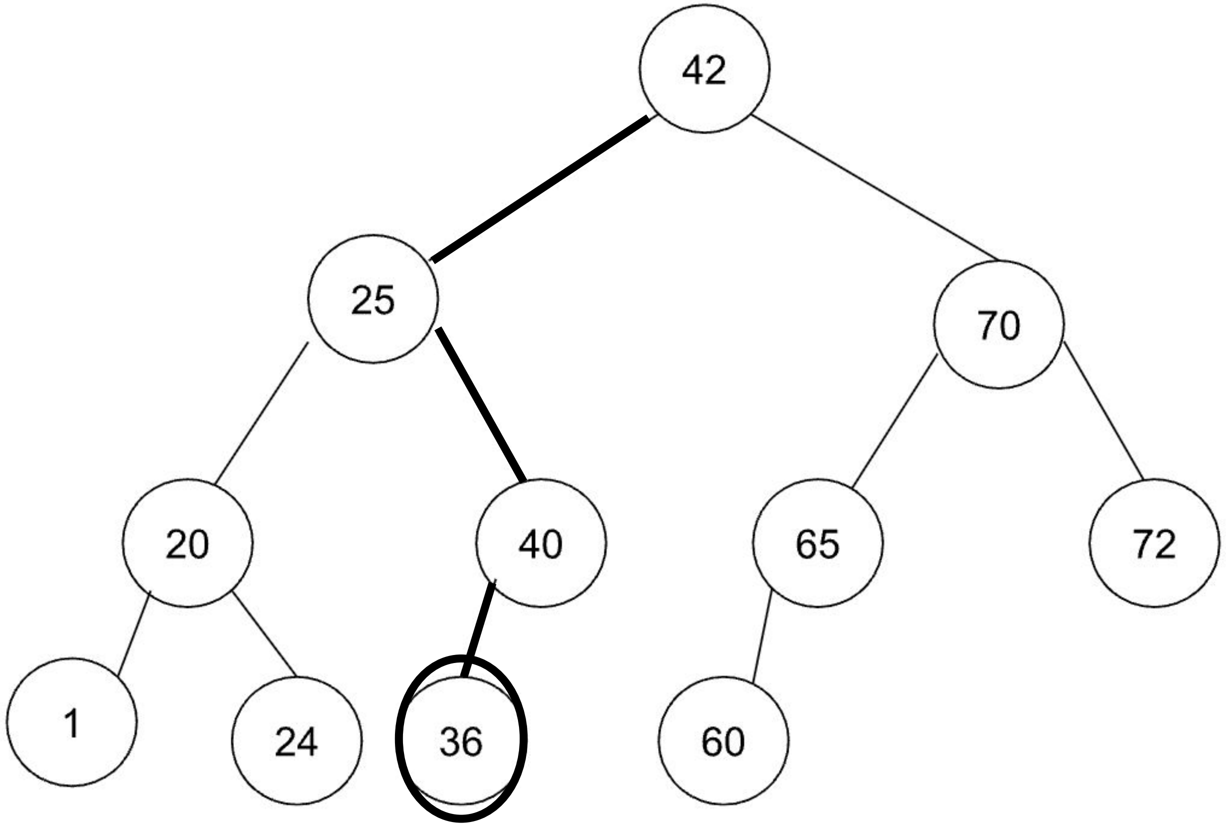

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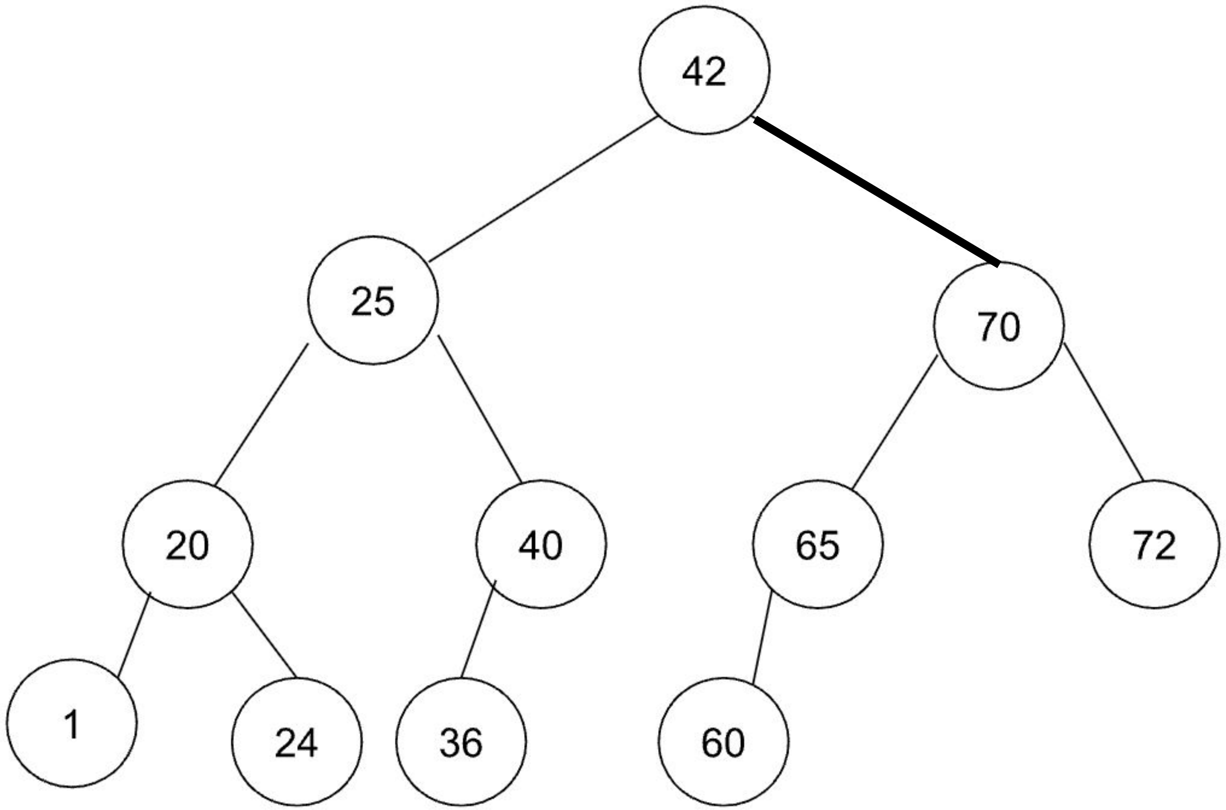

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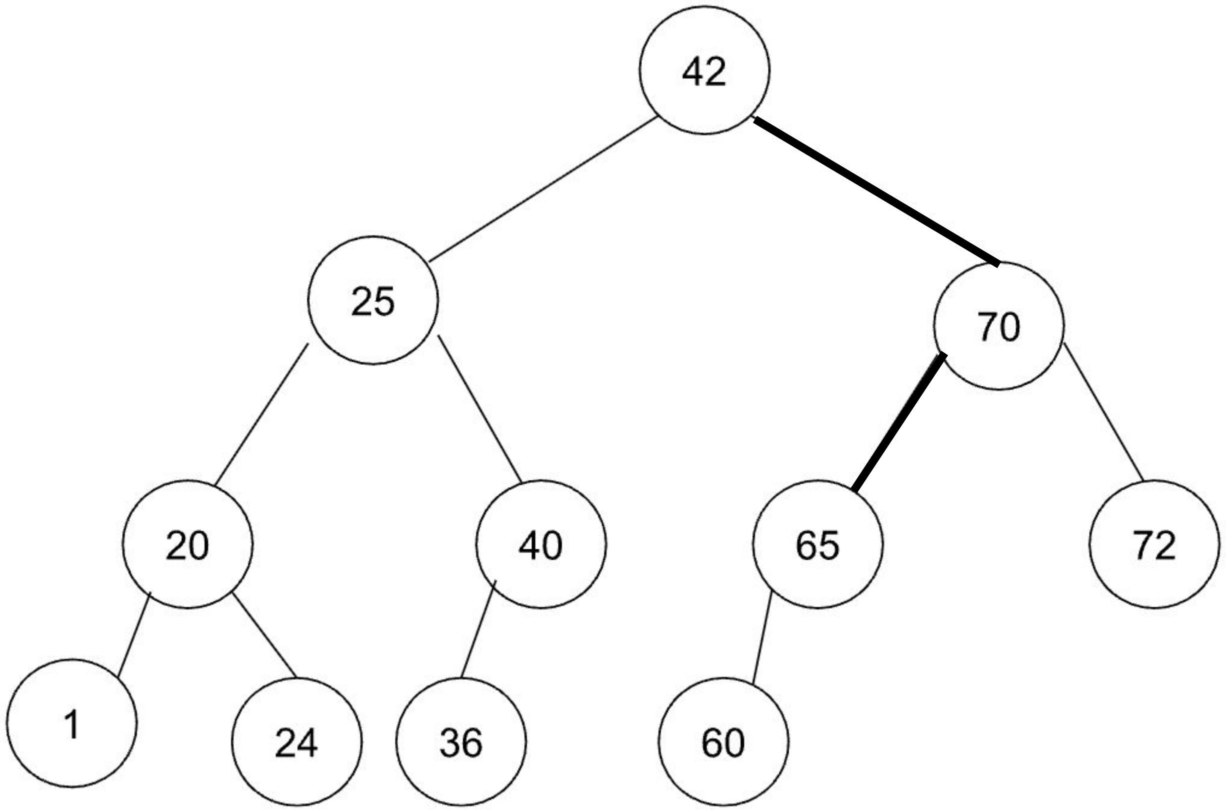

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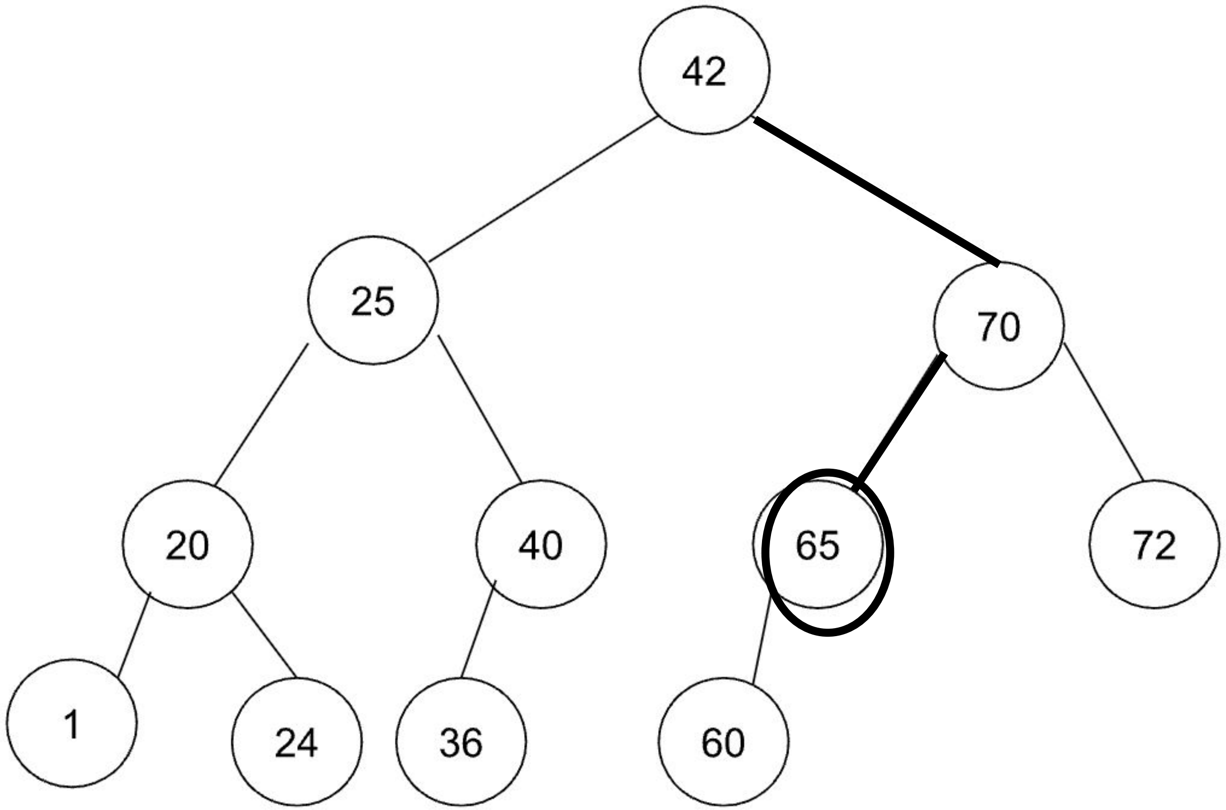

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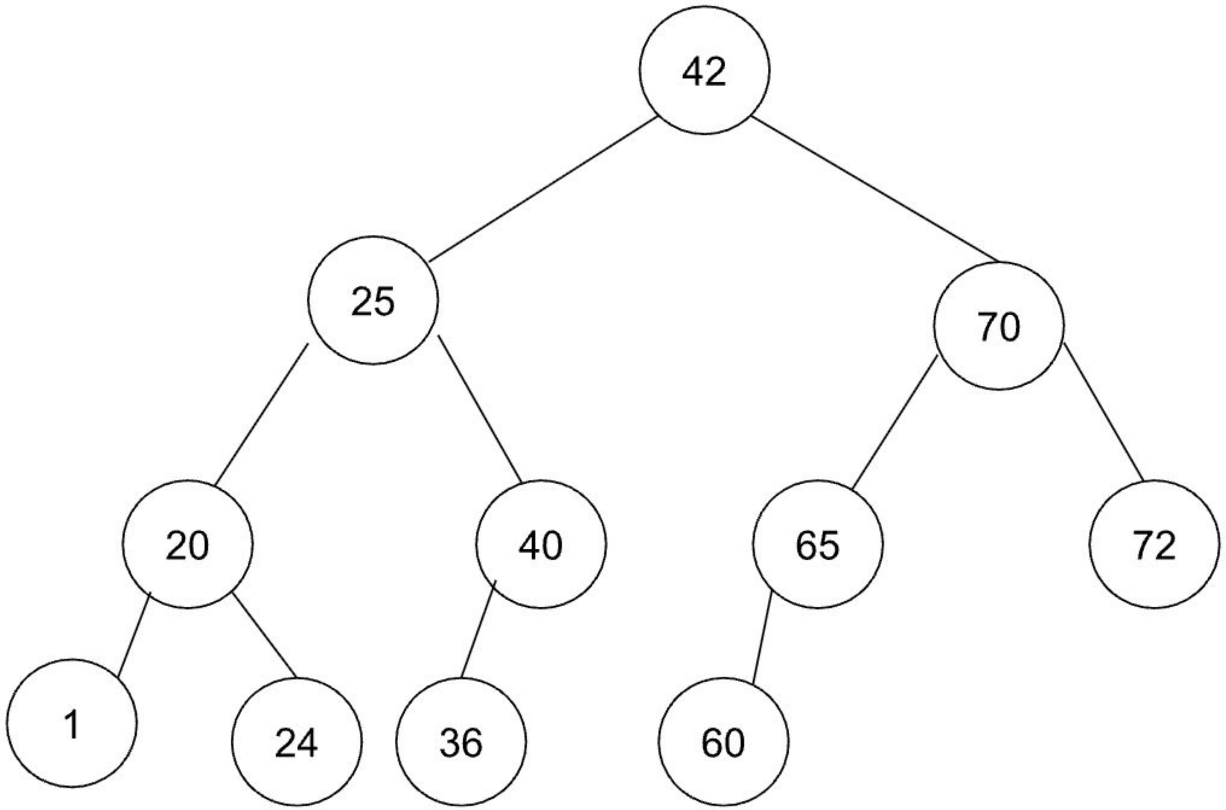

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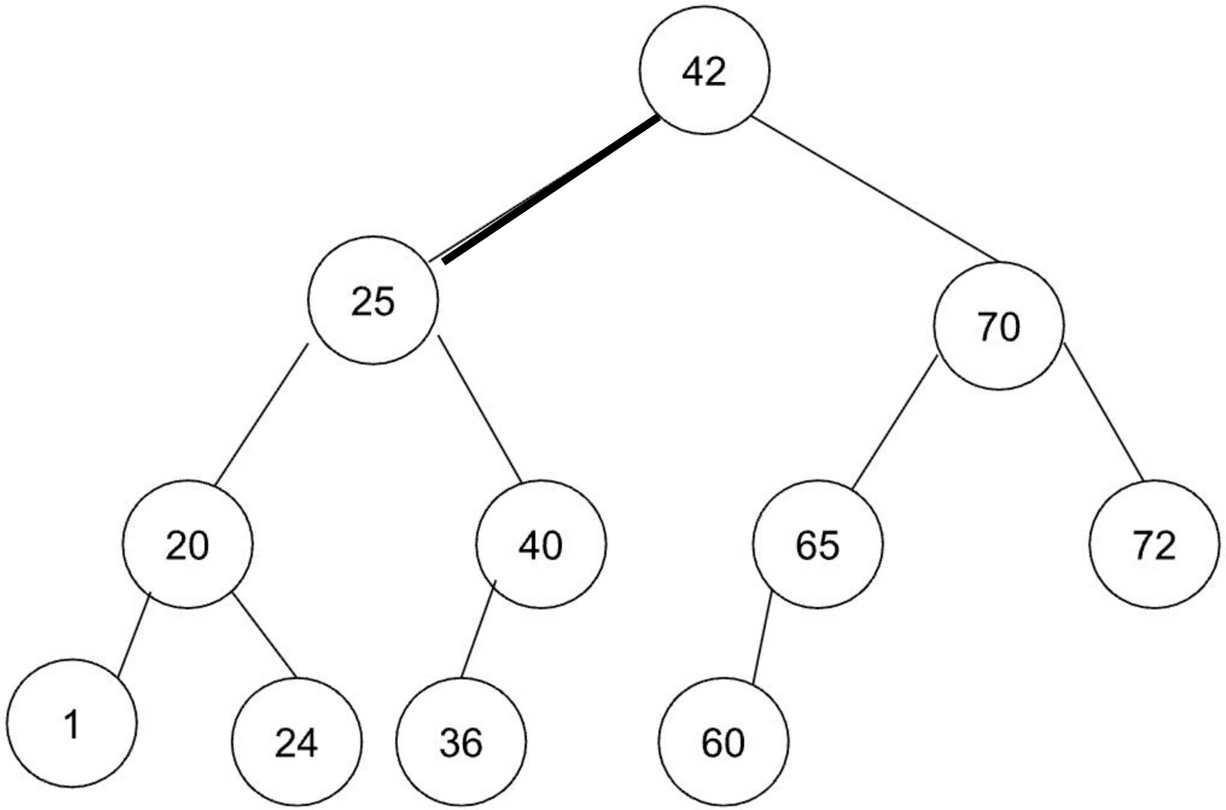

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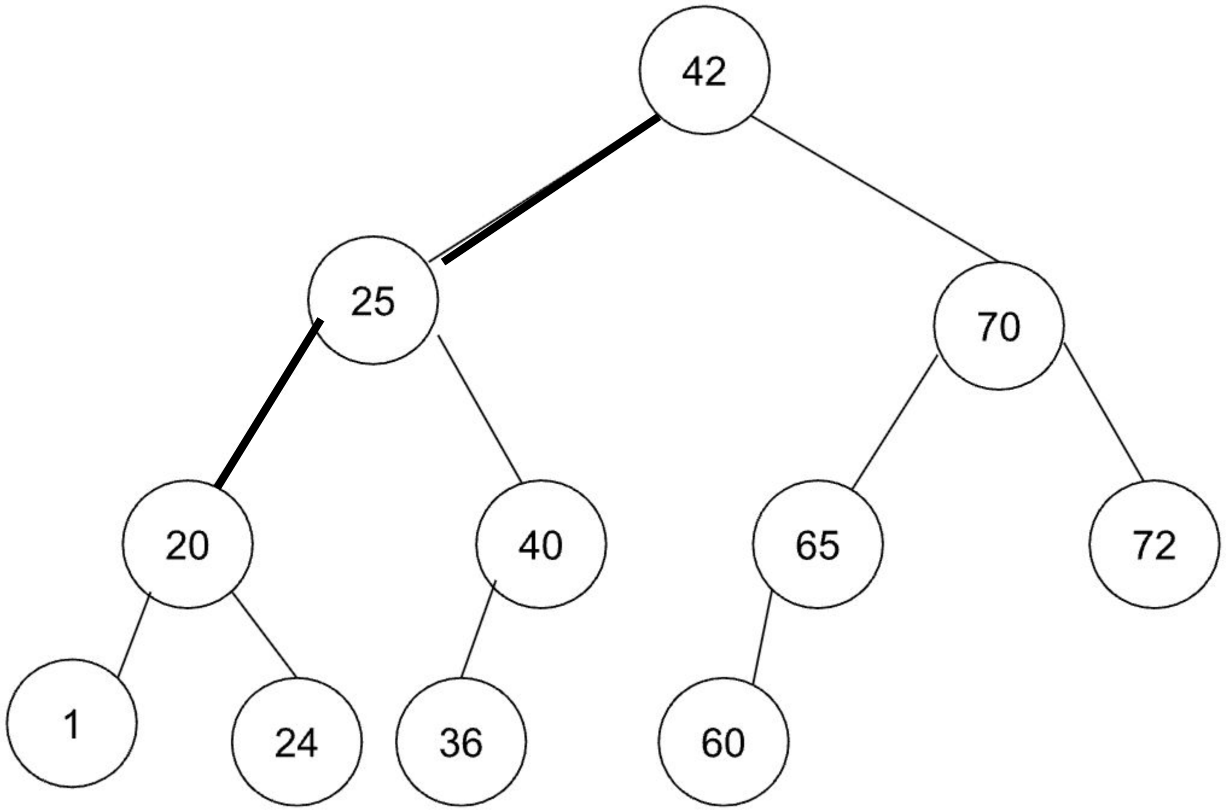

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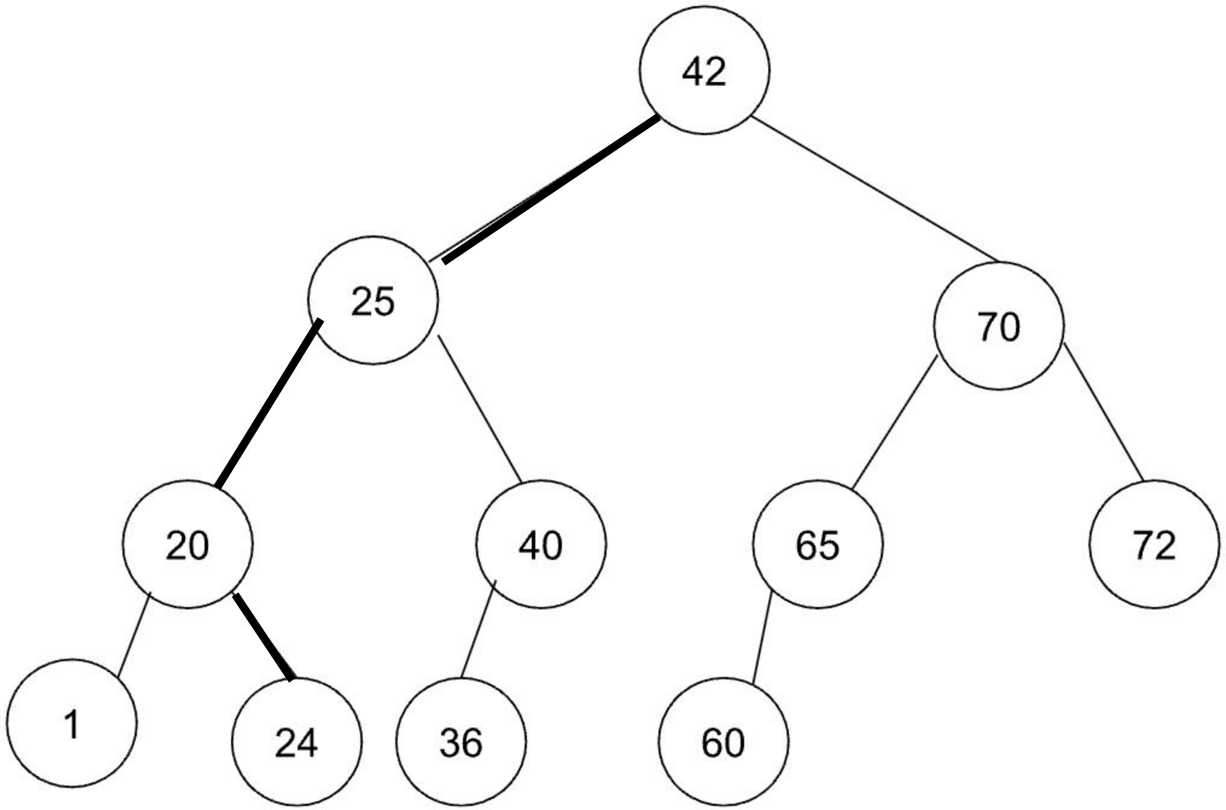
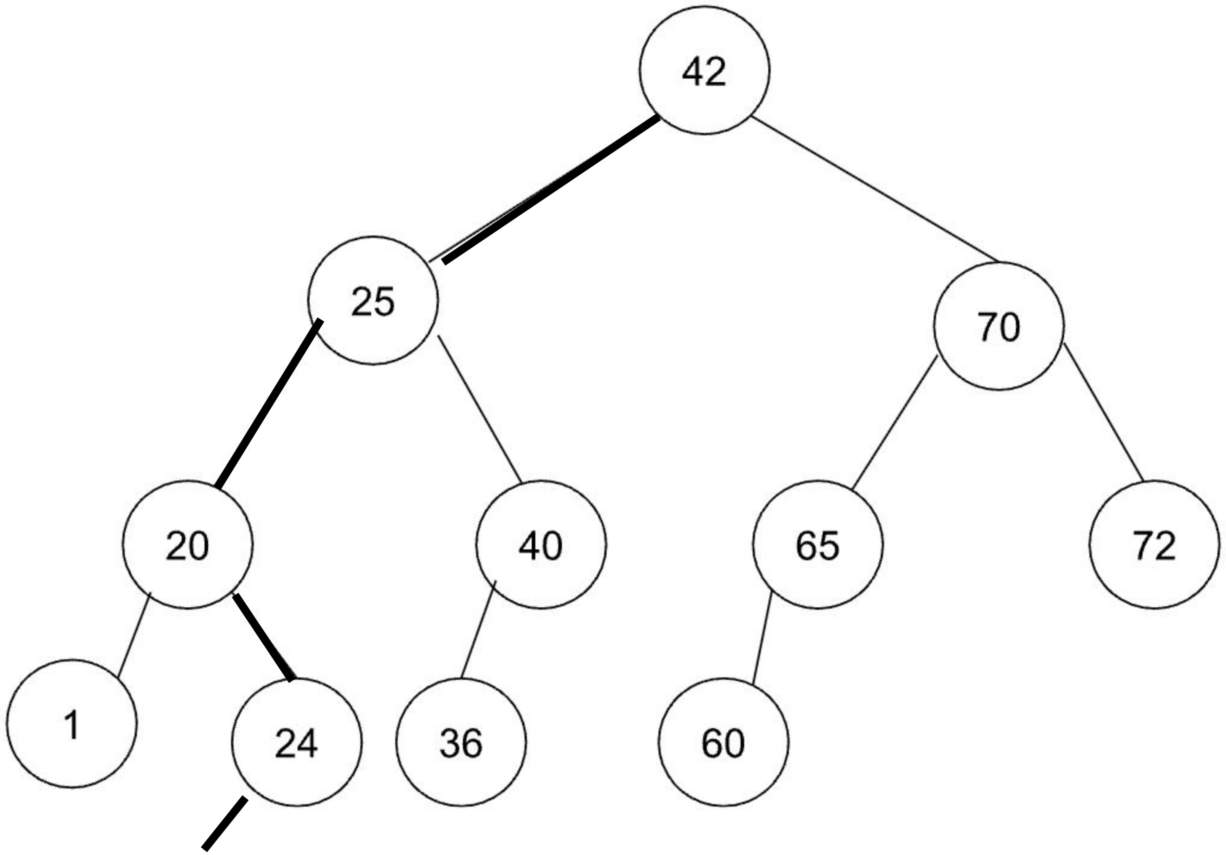


Queries: 23? No!



Queries

- Start walking from `root`.
- If `current` node is:
    - **equal** to target, return *True*;
    - **too large** (> target), follow *left* edge;
    - **too small** (< target), follow *right* edge;
    - **None**, return *False*

```python
def query(self, target):
    """As method of BinarySearchTree."""
    current_node = self.root
    while current_node is not None:
        if current_node.key == target:
            return current_node
        elif current_node.key < target:
            current_node = current_node.right
        else:
            current_node = current_node.left
    return None
```

Exercise: complete

```
def query_recursive(node, target):
    if node is None:
        return False
    if node.key == target:
        ...
    elif ...
        ...
    else ...
        ...
```

Exercise: complete

```
def query_recursive(node, target):
    if node is None:
        return False
    if node.key == target:
        return True
    elif ...
        ...
    else ...
        ...
```

Exercise: complete

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def query_recursive(node, target):
    if node is None:
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    if node.key == target:
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        ...
    else ...
        ...
```

Exercise: complete

```
def query_recursive(node, target):
    if node is None:
        return False
    if node.key == target:
        return True
    elif node.key < target:
        return query_recursive(?, target)
    else ...
        ...
```

Exercise: complete

```
def query_recursive(node, target):
    if node is None:
        return False
    if node.key == target:
        return True
    elif node.key < target:
        return query_recursive(node.right, target)
    else ...
        ...
```

Exercise: complete

```
def query_recursive(node, target):
    if node is None:
        return False
    if node.key == target:
        return True
    elif node.key < target:
        return query_recursive(node.right, target)
    else:
        ...
```

Exercise: complete

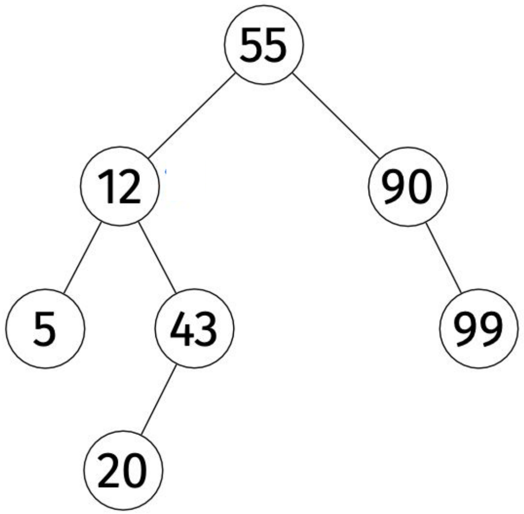
```
def query_recursive(node, target):
    if node is None:
        return False
    if node.key == target:
        return True
    elif node.key < target:
        return query_recursive(node.right, target)
    else:
        return query_recursive(node.left, target)
```

Queries (Recursive)

```python
def query_recursive(node, target):
    """As a 'free function'."""
    if node is None:
        return False

    if node.key == target:
        return node
    elif node.key < target:
        return query_recursive(node.right, target)
    else:
        return query_recursive(node.left, target)
```

Queries, Analyzed



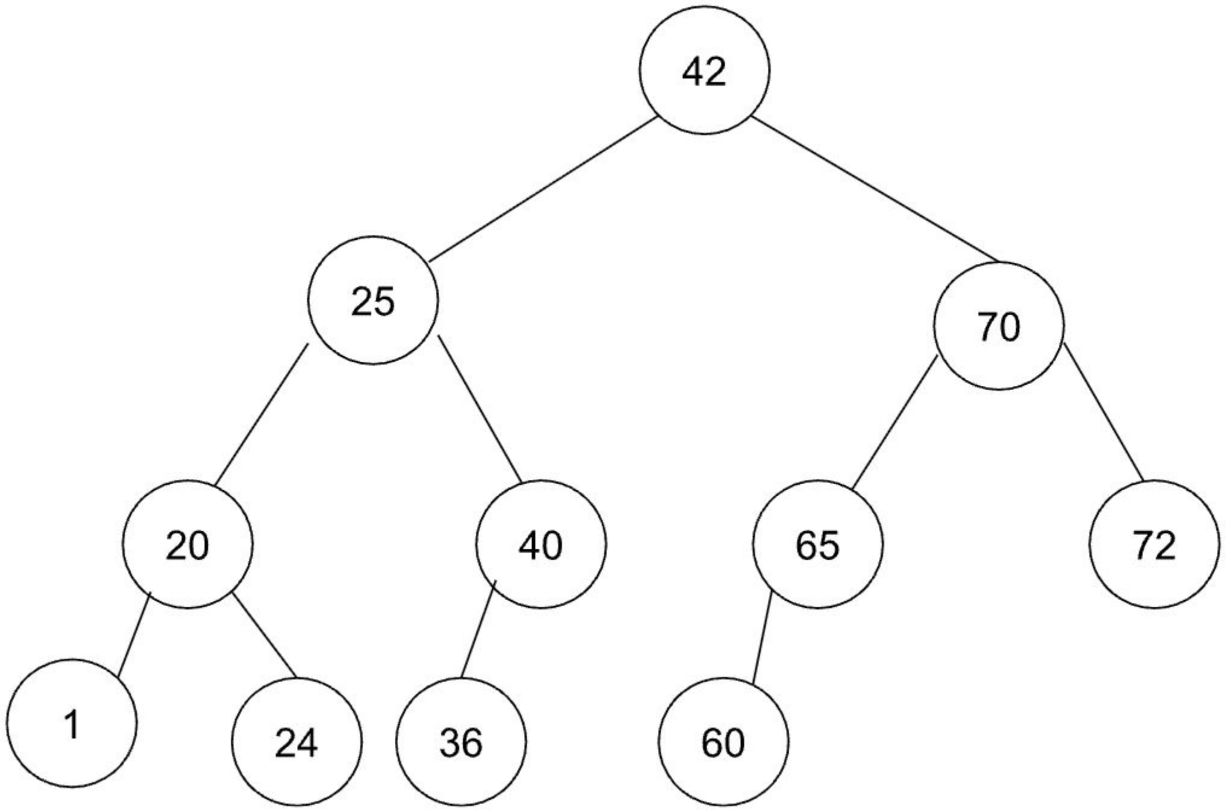
- **Best case**: $\Theta(1)$.
- **Worst case**: $\Theta(h)$, where h is **height** of tree.

Insertion

Insertion

- **Given**: a BST and a new key, k.
- **Modify**: the BST, inserting k.
- Must **maintain** the BST properties.

Insert 23 into the BST.



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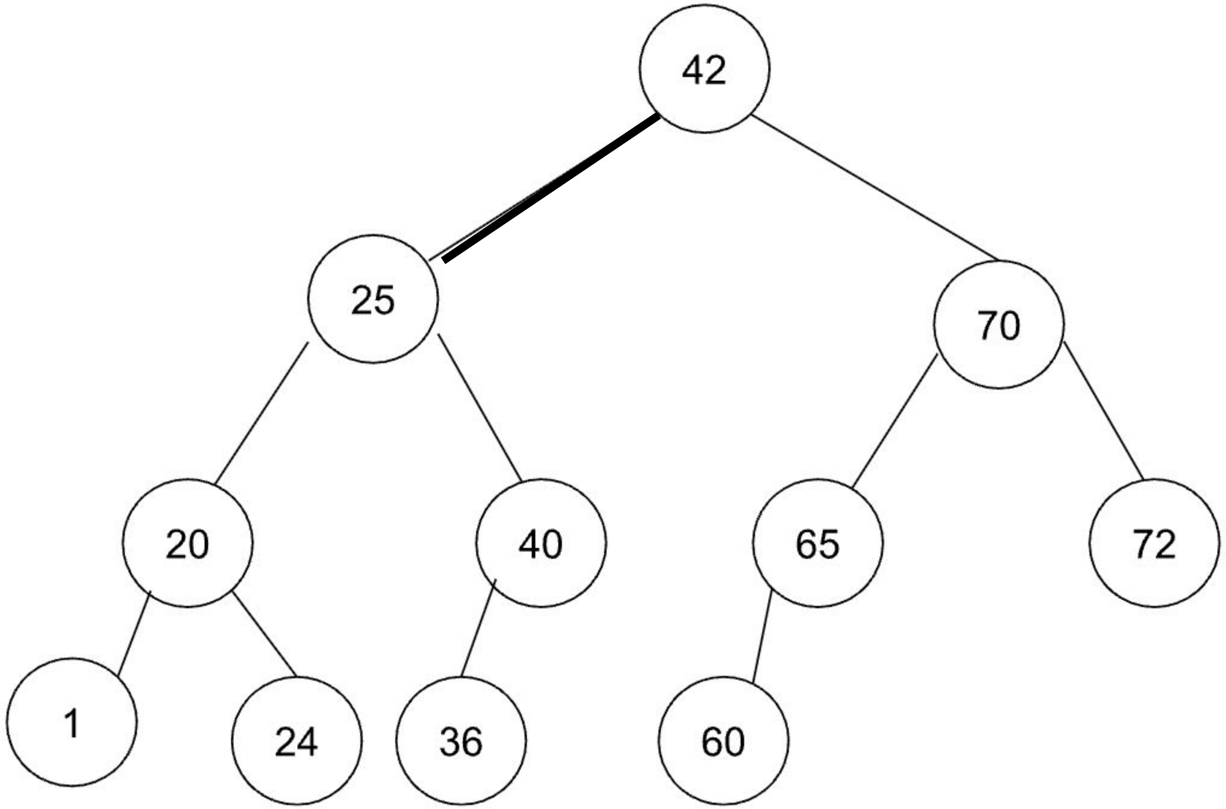

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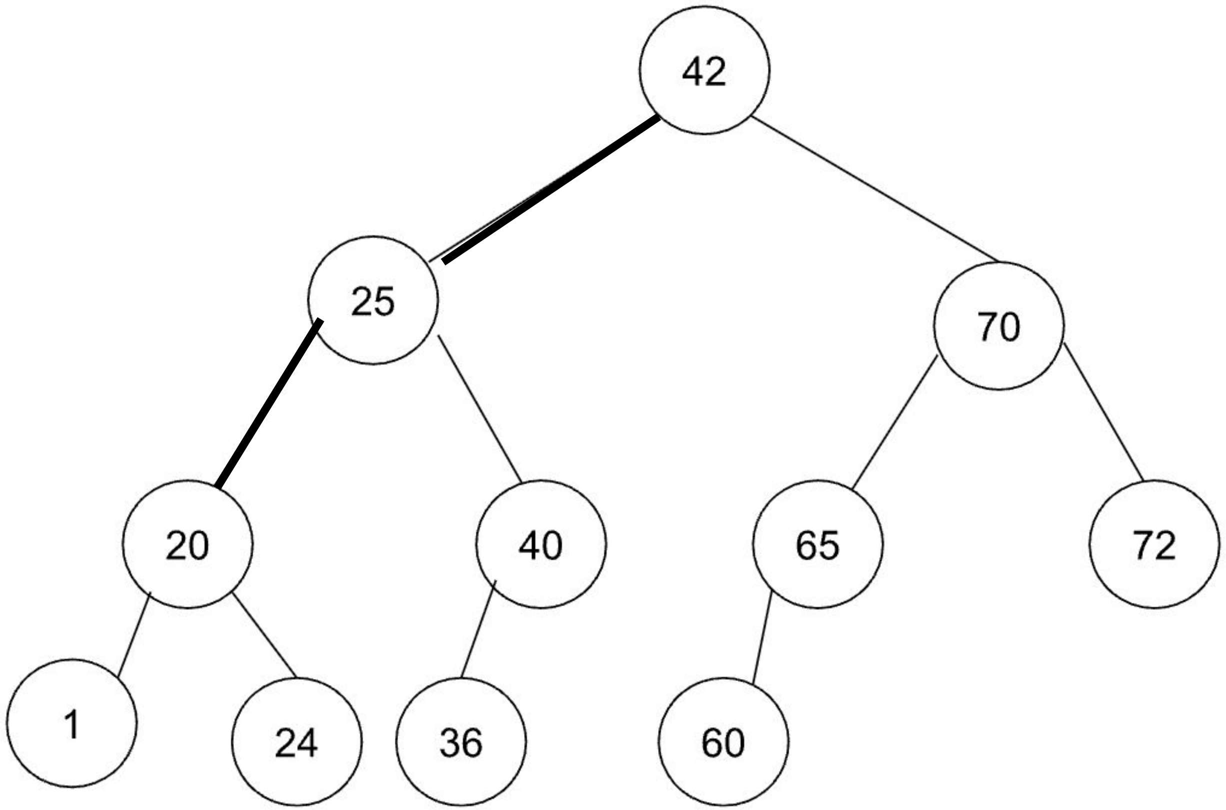

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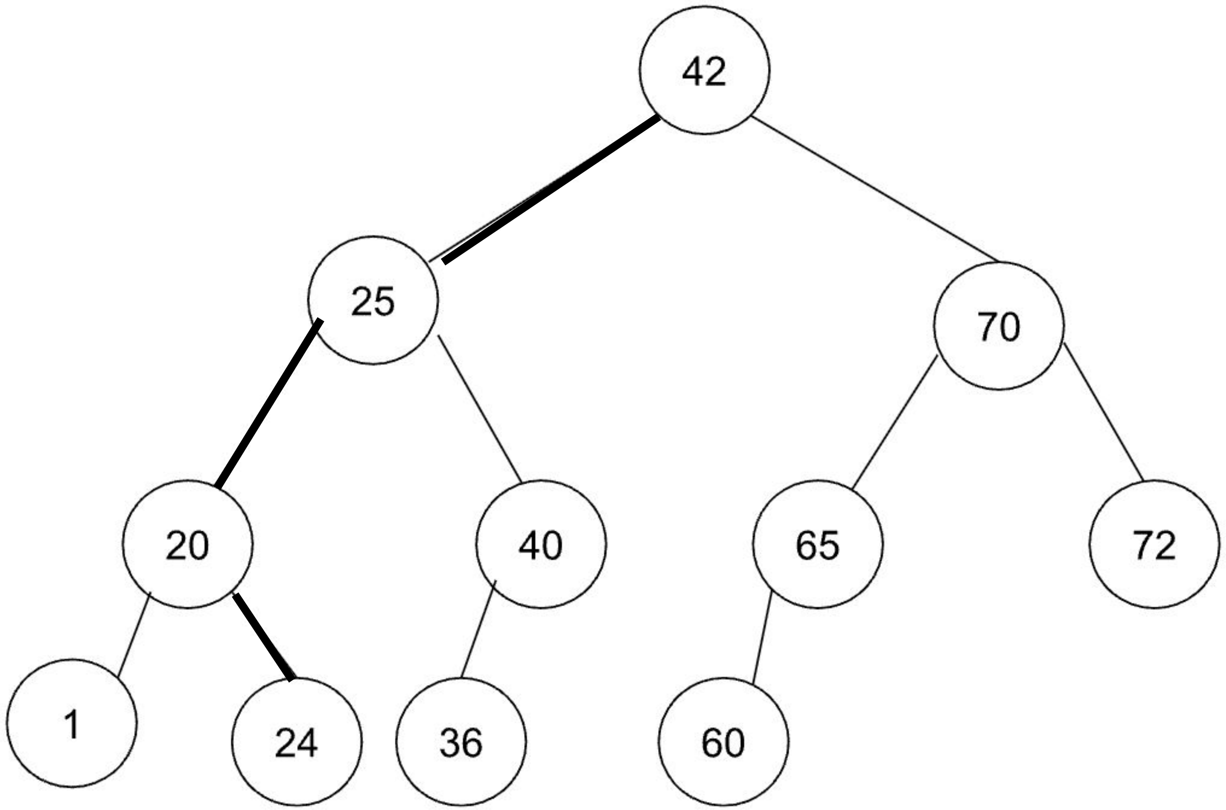

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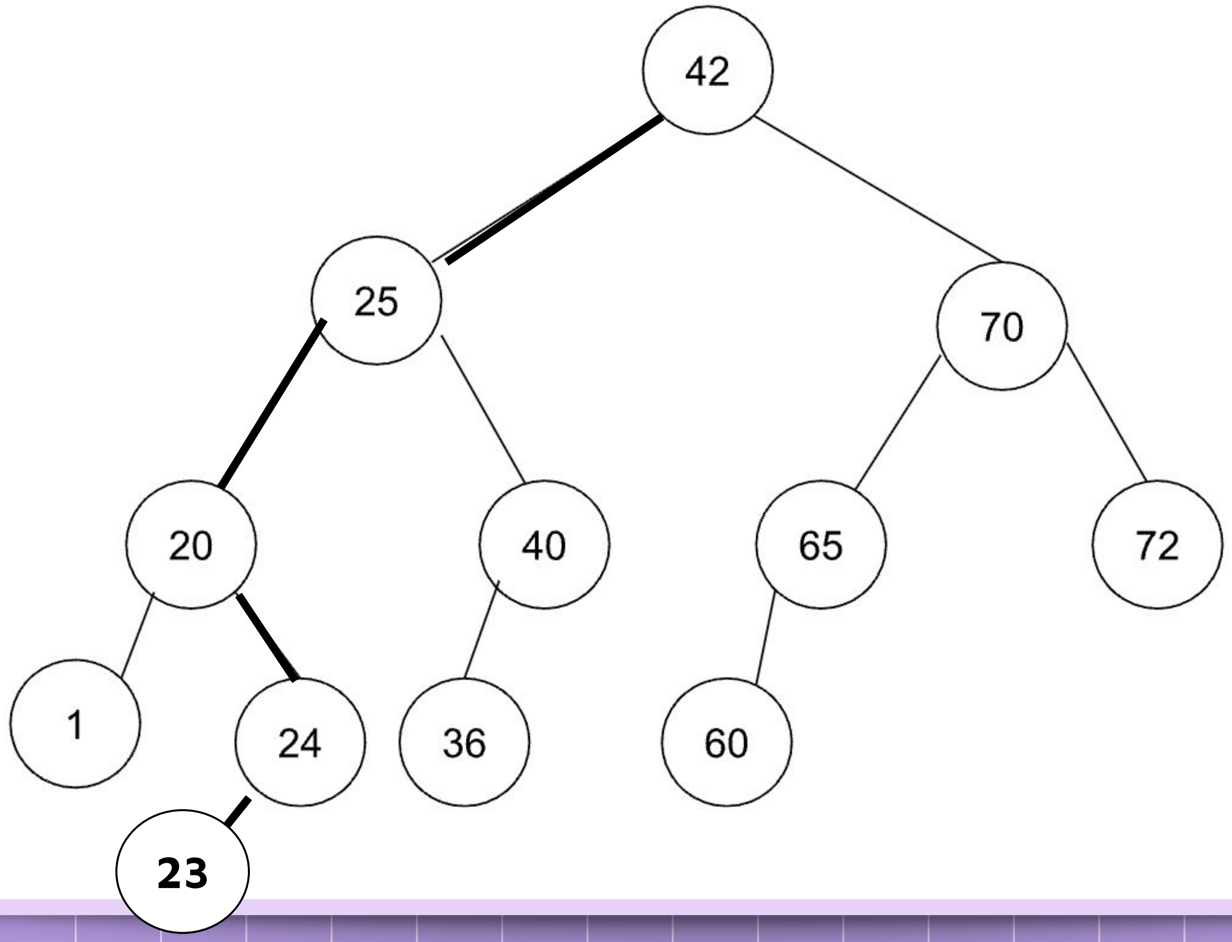


Insertion (The Idea)

- Traverse the tree as in query to find empty spot where new key should go, keeping track of last node seen.
- Create new node; make last node seen the parent, update parent's children.
- *Be careful about inserting into empty tree!*

```python
def insert(self, new_key):
    # assume new_key is unique
    current_node = self.root
    parent = None

    # find place to insert the new node
    while current_node is not None:
        parent = current_node
        if current_node.key < new_key:
            current_node = current_node.right
        else: # current_node.key > new_key
            current_node = current_node.left

    # create the new node
    new_node = Node(key=new_key, parent=parent)

    # if parent is None, this is the root. Otherwise, update the
    # parent's left or right child as appropriate
    if parent is None:
        self.root = new_node
    elif parent.key < new_key:
        parent.right = new_node
    else:
        parent.left = new_node
```

Insertion, Analyzed

- Worst case: $\Theta(h)$, where h is **height** of tree.

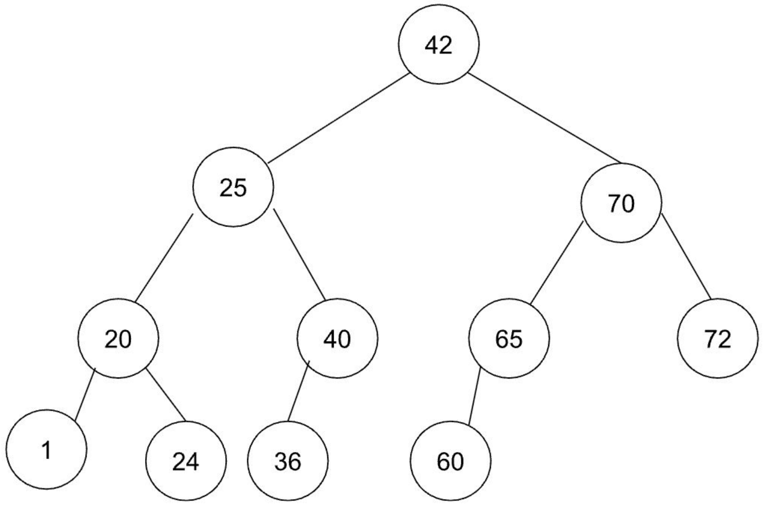
Main Idea

Querying and **insertion** take $\Theta(h)$ time in the worst case, *where h is the height of the tree.*

Balanced and Unbalanced BSTs

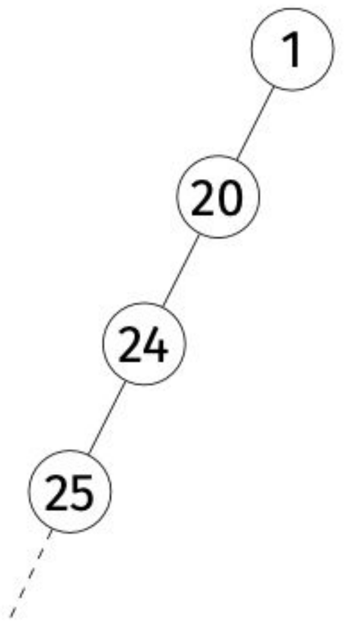
Binary Tree Height

- In case of very **balanced** tree, $h = \Theta(\log n)$.
  - Query, insertion take worst case $\Theta(\log n)$ time in a **balanced** tree.



Binary Tree Height

- In the case of very **unbalanced** tree, $h = \Theta(n)$.
  - Query, insertion take worst case $\Theta(n)$ time in a **unbalanced** tree.



Unbalanced Trees

- Occurs if we insert items in (close to) sorted or reverse sorted order.
- This is a **common** situation.

Example

- **Flight schedules** — airline departure/arrival times are often listed in *chronological* order: 7:00, 8:00, 10:00, 13:00 ….
- Insert 7, 8, 10, 13 (in that order).

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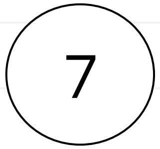

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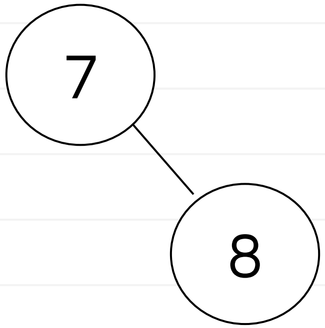

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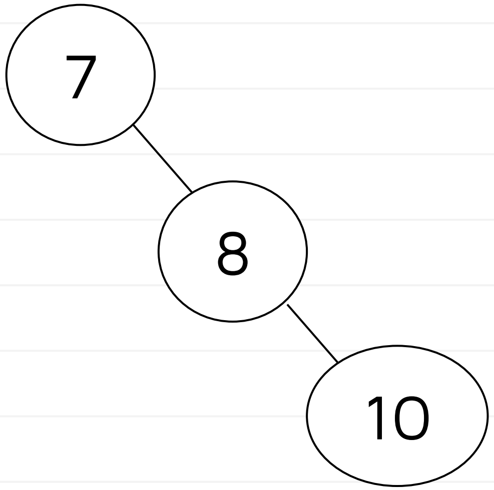

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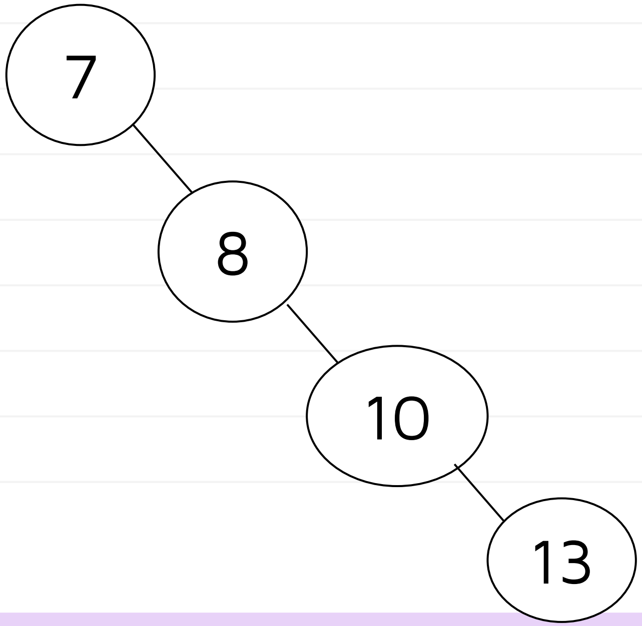
- **Flight schedules** — airline departure/arrival times are often listed in *chronological* order: 7:00, 8:00, 10:00, 13:00 ....
- Insert 7, 8, 10, 13 (in that order).



Time Complexities

- query $\Theta(h)$
- insertion $\Theta(h)$

- Where h is height, and $h = \Omega(\log n)$ and $h = O(n)$.

Time Complexities: (Balanced)

- query $\Theta(\log n)$
- insertion $\Theta(\log n)$

- Where h is height, and $h = \Omega(\log n)$ and $h = O(n)$.

Worst Case Time Complexities (Unbalanced)

query $\Theta(n)$
insertion $\Theta(n)$

- The worst case is **bad**.
  - Worse than using a sorted array!
- The worst case is **not rare.**

Main Idea

The operations take **linear time** in the worst case unless we can somehow ensure that the tree is **balanced**.

Self-Balancing Trees

Self-Balancing Trees

- There are variants of BSTs that are **self-balancing**.
  - Red-Black Trees, AVL Trees, etc.
- Quite complicated to implement correctly.
- But their height is **guaranteed** to be ~ log n.
- **So insertion, query take $\Theta(\log n)$ in worst case.**

Quick Visualization: B-trees

Warning

If asked for the time complexity of a BST operation, **be careful**!

A common mistake is to say that insertion/query are $\Theta(\log n)$ without being told that the tree is balanced.

Main Idea

In general, insertion/query take $\Theta(h)$ time in worst case.

- If tree is balanced, $h = \Theta(\log n)$, so they take $\Theta(\log n)$ time.
- If tree is badly unbalanced, $h = O(n)$, and they can take $O(n)$ time.

Augmenting BSTs

Modifying BSTs

- Perhaps more than most other data structures, BSTs must be modified (**augmented**) to solve unique problems.

Order Statistics

- Given n numbers, the kth order statistic is the kth *smallest* number in the collection.

Dynamic Set, Many Order Statistics

- `Quickselect` finds any order statistic in *linear* expected time.
- This is efficient for a **static set**.
- **Inefficient** if set is *dynamic*.

Goal

- Create a **dynamic** set data structure that supports fast computation of **any** order statistic.

BST Solution

- For each node, keep attribute `.size`, containing #of nodes in subtree rooted at current node:

- **Property**:

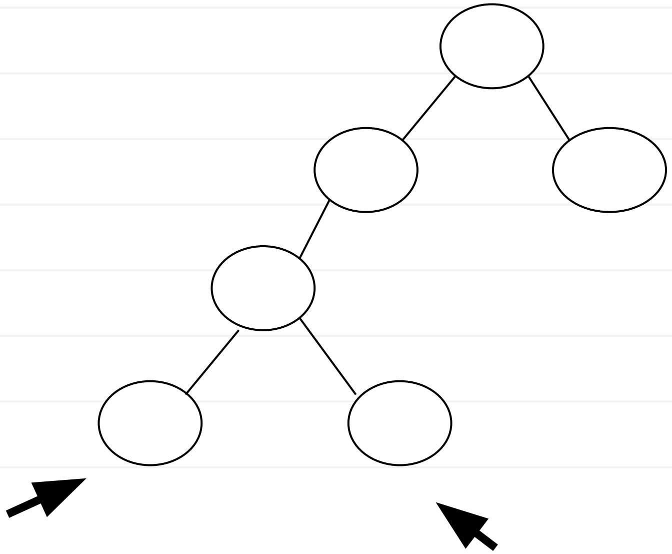
```
x.size = x.left.size + x.right.size + 1
```

If a left or right child doesn't exist, consider its size zero.

BST Solution

Property:

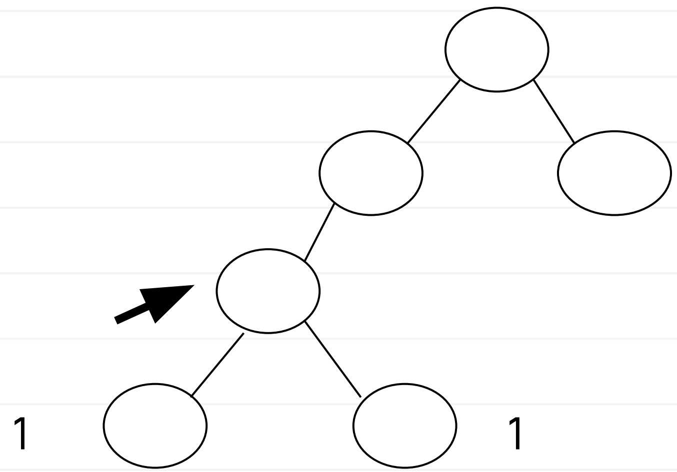
`x.size = x.left.size + x.right.size + 1`



BST Solution

Property:

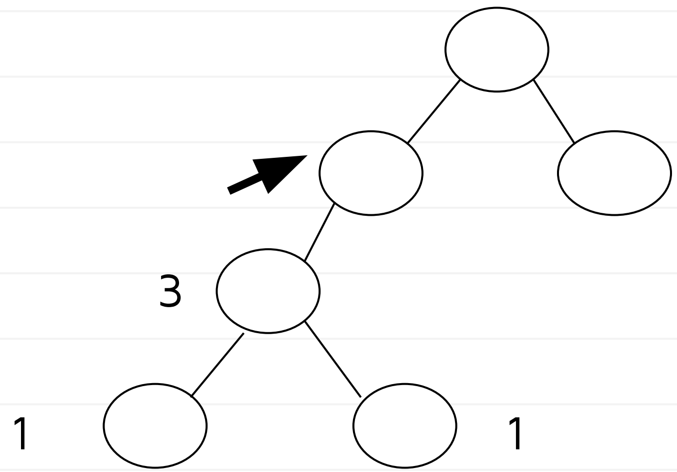
```
x.size = x.left.size + x.right.size + 1
```



BST Solution

Property:

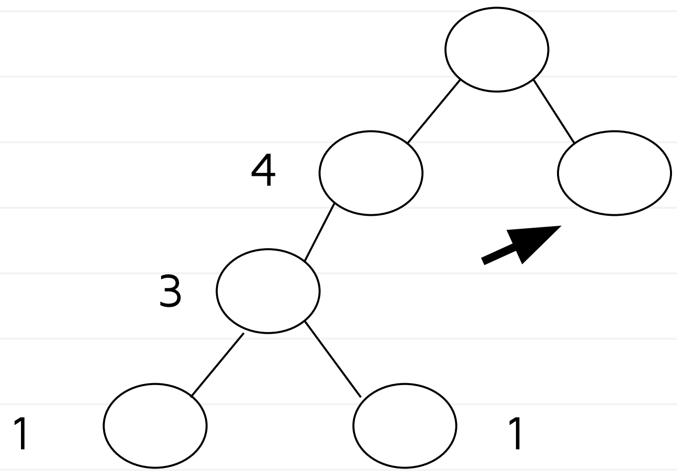
```
x.size = x.left.size + x.right.size + 1
```



BST Solution

Property:

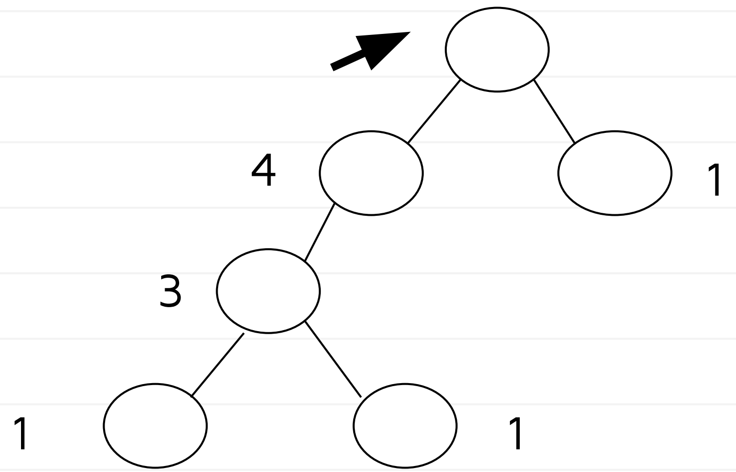
```
x.size = x.left.size + x.right.size + 1
```



BST Solution

Property:

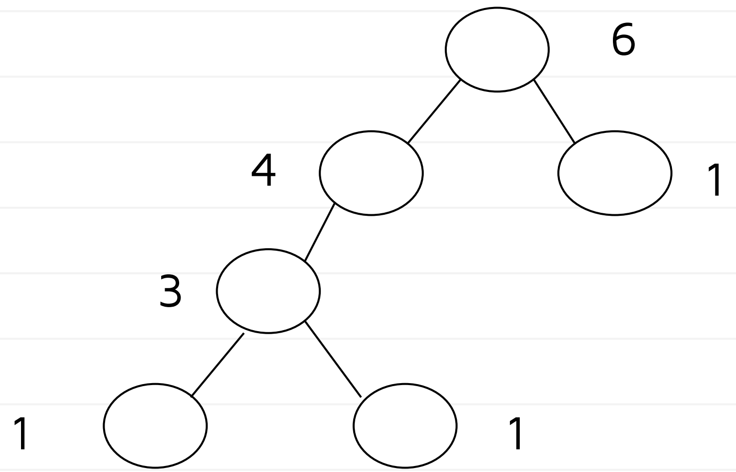
```
x.size = x.left.size + x.right.size + 1
```



BST Solution

Property:

```
x.size = x.left.size + x.right.size + 1
```



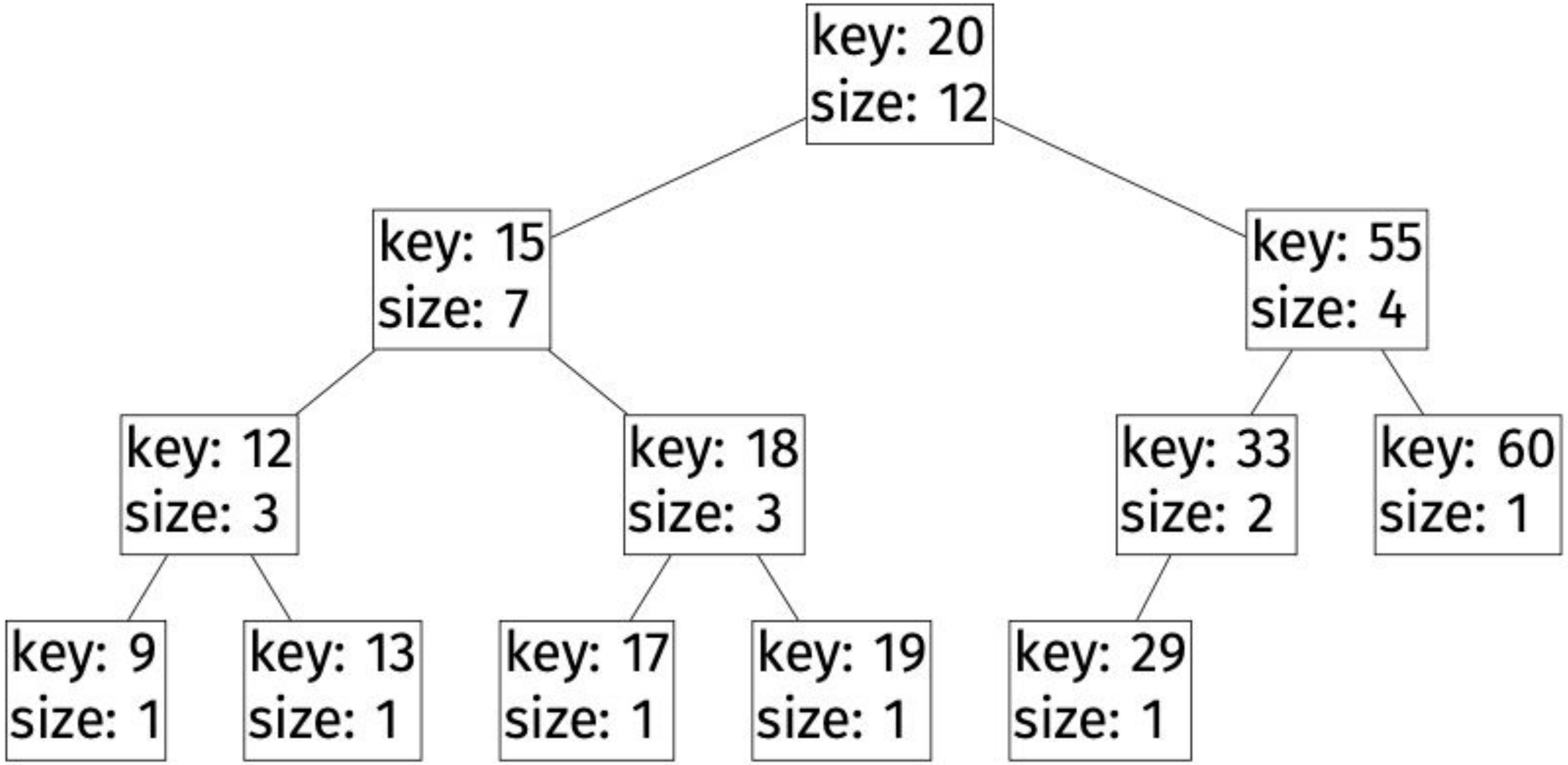
Computing Sizes

```
def add_sizes_to_tree(node):
    if node is None:
        return 0
    left_size = add_sizes_to_tree(node.left)
    right_size = add_sizes_to_tree(node.right)
    node.size = left_size + right_size + 1
    return node.size
```

Note

- Also need to maintain `size` upon inserting a node.

Computing Order Statistics: 8th? 2nd? 12th



Augmenting Data Structures

- This is just **one** example, but many more.
- Understanding how BSTs work is key to augmenting them.

Thank you!

Do you have any questions?

CampusWire!