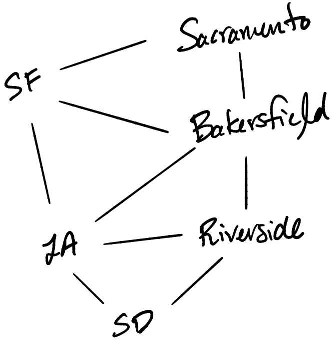
DSC 40B

Theoretical Foundations II

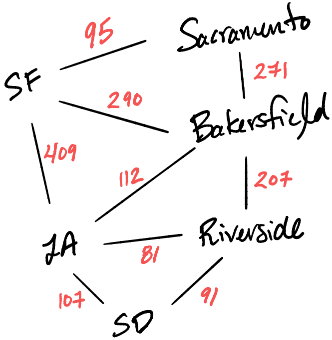
Lecture 14 | Part 1

Shortest Paths in Weighted Graphs

Google Maps



Google Maps



Sacramento
SF
Bakersfield
LA
Riverside
SD
95
271
290
409
112
207
81
107
91

Weighted Graphs

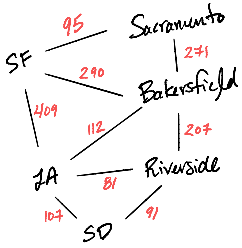
An **edge weighted graph** $G = (V, E, \omega)$ is a triple where (V, E) is a graph and $\omega : E \to \mathbb{R}$ maps each edge to a **weight**.

- Can be directed or undirected.
- In general, weights can be positive, negative, zero.
- Many uses, such as representing **metric spaces**.

Path Lengths

The **length** of a path in a **weighted graph** (usually) refers to the total weight of all edges in the path.

Example: (SD, Riverside, Bakersfield, SF)

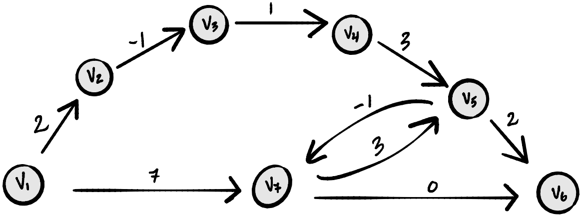


Shortest Paths

- A **shortest path** between u and v is a path between u and v with minimum length.
  - In other words, minimum total weight.

Example

What is the shortest path from v_1 to v_6?



Path:
Length:

Today (and next time)

How do we find shortest paths in weighted graphs?

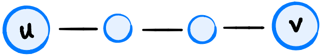
Idea #0

- Does BFS work?
  - **No**, not really. Only if all weights are the same.

- Can we “convert” a weighted graph to an unweighted one?

Idea #0

Idea #0



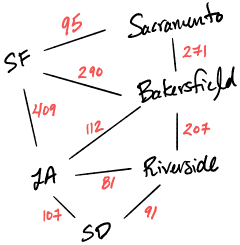
Idea #0

- Step 1: “Convert” weighted graph to unweighted one with dummy nodes.

- Step 2: Call BFS on this new graph.

Idea #0

- **Very inefficient** for large weights.



- What if edge weights are floats, or negative?

Ideas #1 and #2

- We'll look at two algorithms: **Bellman-Ford** and **Dijkstra's**.

  INPUT: weighted graph, source vertex s.
  OUTPUT: shortest paths from s to every other node.

- Both work by:
  - keeping track of shortest known path (**estimates**).
  - iteratively **updating** these until they're correct.

Shortest Path Estimates

- B-F and Dijkstra's keep track of the shortest paths found *so far*; we call these the **estimated shortest paths**.
- For each node *u*, remember *u*'s:
  - predecessor in estimated shortest path;
  - distance from source *s* in estimated shortest path.
- **Key**: estimated distance will always be ≥ actual distance.

Updates

- Both algorithms work by iteratively updating their estimates.

- On each iteration, consider a new edge (u, v). Ask: is the best known shortest path from

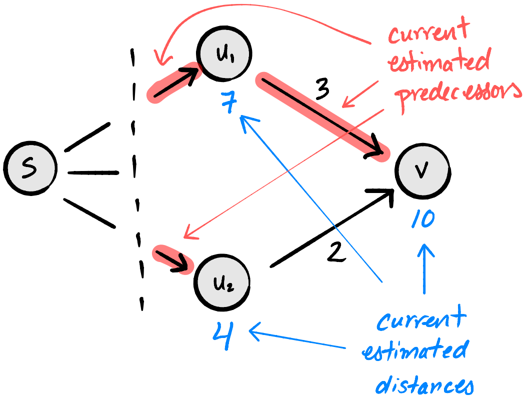
  $$\text{source} \to \cdots \to u \to v$$

  shorter than the best known shortest path from

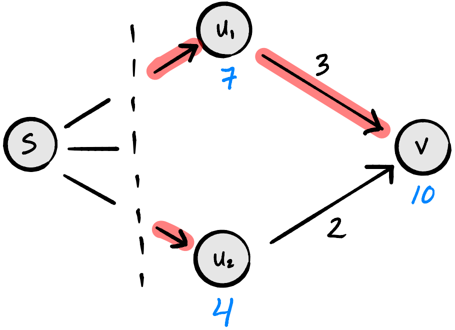
  $$\text{source} \to \cdots \to \texttt{predecessor[v]} \to v?$$

- If it is, we have discovered a shorter path to v.

Example: Updating (u_2, v):



Example: Updating (u_2, v):

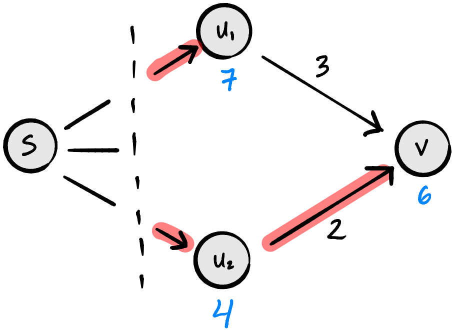


estimated length of $s \to u_2 \to v$

$= (\text{estimated length of } s \to u_2) + \omega(u_2, v)$

$= 4 + 2 = 6 < 10$

Example: After Updating (u_2, v):



A shorter path has been found.

Updating, in Code

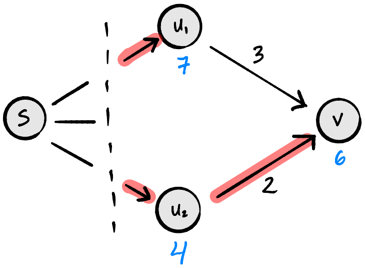
- Let:
  - `est` be a dictionary of estimated shortest path distances.
  - `predecessor` be a dictionary of estimated shortest path predecessors.
  - `weights` be a function which returns edge weights.

Updating, in Code

```
def update(u, v, weights, est, predecessor):
    """Update edge (u,v)."""
    if est[v] > est[u] + weights(u,v):
        est[v] = est[u] + weights(u,v)
        predecessor[v] = u
        return True
    else:
        return False
```

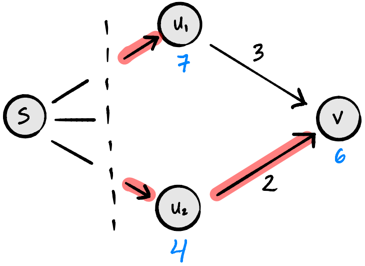
- Time complexity: ______

When does an update discover a shortest path?



- Suppose updating (u_2, v) finds a shorter path to v.
- True or False: the actual shortest path must go through u_2.

When does an update discover a shortest path?



- Suppose updating (u_2, v) finds a shorter path to v.
- True or False: the actual shortest path must go through u_2.
- **False**: we might later discover a better path to u_1.

When does an update discover the shortest path?

- Let (u, v) be an edge.
- Suppose:
  - the actual shortest path to u has been found;
  - the actual shortest path to v goes through (u, v).
- Then after updating (u, v), the estimated shortest path to v is correct.

DSC 40B

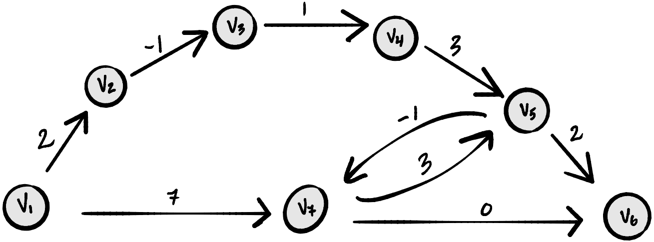
Theoretical Foundations II

Lecture 14 | Part 2

Bellman-Ford

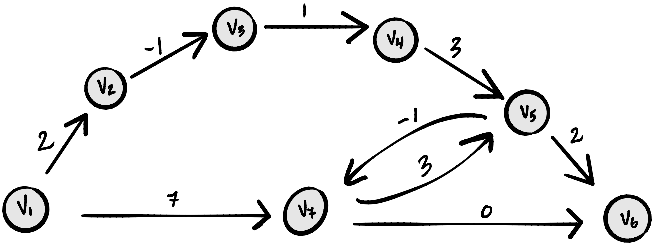
Intuition

- Shortest paths that have many edges are "harder" to discover.
  - May require many updates.
- Shortest paths that have few edges are "easier" to discover.
- Once we've discovered all of the shortest paths with few edges, it makes it easier to discover the shortest paths with more edges.



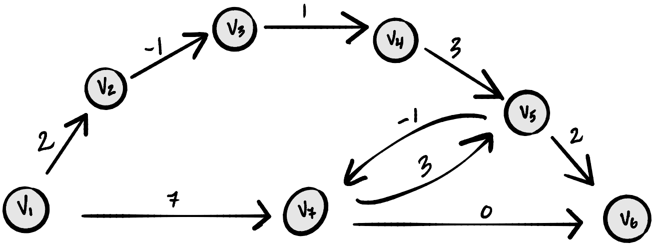
V1
V2
V3
V4
V5
V6
V7
2
-1
1
3
2
-1
3
7
0

Updating All Edges



- Suppose we update all of the edges, one by one.
- Then all nodes whose shortest path from *s* has only **one** edge are **guaranteed** to be estimated correctly.

Updating All Edges



- Suppose we update all of the edges again.
- Then all nodes whose shortest path from s has at most **two** edges are **guaranteed** to be estimated correctly.

Loop Invariant

- One iteration: update all edges in arbitrary order.
- Loop invariant: After α iterations, all nodes whose shortest path from s has $\leq \alpha$ edges are guaranteed to be estimated correctly.

The Bellman-Ford Algorithm

```
def bellman_ford(graph, weights, source):
    """Assume graph is directed."""
    est = {node: float('inf') for node in graph.nodes}
    est[source] = 0
    predecessor = {node: None for node in graph.nodes}

    for i in range(?):
        for (u, v) in graph.edges:
            update(u, v, weights, est, predecessor)

    return est, predecessor
```

Bellman-Ford

- Claim: each node must have a shortest path which is simple[1].

- The most edges a simple path can have is $|V| - 1$

- Idea of Bellman-Ford: iteratively update all edges, repeat $|V| - 1$ times.

[1] Edge case: cycles of weight zero.

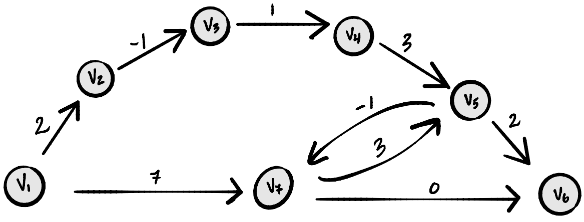
The Bellman-Ford Algorithm

```
def bellman_ford(graph, weights, source):
    """Assume graph is directed."""
    est = {node: float('inf') for node in graph.nodes}
    est[source] = 0
    predecessor = {node: None for node in graph.nodes}

    for i in range(len(graph.nodes) - 1):
        for (u, v) in graph.edges:
            update(u, v, weights, est, predecessor)

    return est, predecessor
```

Example

Suppose `graph.edges` returns edges in following order:

$$(v_3, v_4), (v_1, v_2), (v_2, v_3), (v_7, v_6), (v_5, v_7),$$
$$(v_7, v_5), (v_4, v_5), (v_5, v_6), (v_1, v_7)$$



Time Complexity

```
def bellman_ford(graph, weights, source):
    """Assume graph is directed."""
    est = {node: float('inf') for node in graph.nodes}
    est[source] = 0
    predecessor = {node: None for node in graph.nodes}

    for i in range(len(graph.nodes) - 1):
        for (u, v) in graph.edges:
            update(u, v, weights, est, predecessor)

    return est, predecessor
```

- Setup: ____________ time
- Each update takes ____________ time
- There are exactly ____________________________ updates
- Total time complexity: ____________

DSC 40B

Theoretical Foundations II

Lecture 14 | Part 3

Early Stopping and Negative Cycles

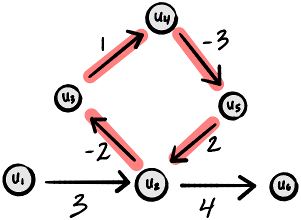
Early Stopping

- B-F may not need to run for $|V| - 1$ iterations.
- If no predecessors change, we can break:

```
def bellman_ford(graph, weights, source):
    """Early stopping version."""
    est = {node: float('inf') for node in graph.nodes}
    est[source] = 0
    predecessor = {node: None for node in graph.nodes}

    for i in range(len(graph.nodes) - 1):
        any_changes = False
        for (u, v) in graph.edges:
            changed = update(u, v, weights, est, predecessor)
            any_changes = changed or any_changes
        if not any_changes:
            break
    return est, predecessor
```

Negative Cycles

- A **negative cycle** is a cycle whose total edge weight is negative:



- If a graph has a negative cycle, (some) shortest paths are **not well defined**.

Detecting Negative Cycles

- If graph **does not have** negative cycles, estimated distances eventually stop changing (after at most $|V| - 1$ iterations).

- If graph **has** negative cycles, estimated distances always decrease.

- To detect them: run a $|V|$th iteration; if distances change, a negative cycle exists.

Detecting Negative Cycles

```
def bellman_ford(graph, weights, source):
    """Early stopping version, detects negative cycles."""
    est = {node: float('inf') for node in graph.nodes}
    est[source] = 0
    predecessor = {node: None for node in graph.nodes}

    for i in range(len(graph.nodes)):
        any_changes = False
        for (u, v) in graph.edges:
            changed = update(u, v, weights, est, predecessor)
            any_changes = changed or any_changes
        if not any_changes:
            break
    # this will be True if negative cycles exist
    contains_negative_cycles = any_changes
    return est, predecessor, contains_negative_cycles
```