DSC 40B
Theoretical Foundations II

Lecture 12 | Part 1

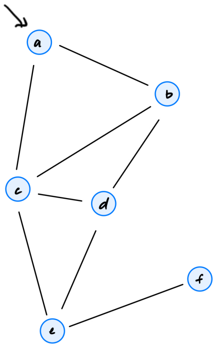
Warmup: Aggregate Analysis

Time Complexity

```
def full_bfs(graph):
    status = {node: 'undiscovered' for node in graph.nodes}
    for node in graph.nodes:
        if status[node] == 'undiscovered'
            bfs(graph, node, status)


def bfs(graph, source, status=None):
    """Start a BFS at `source`."""
    if status is None:
        status = {node: 'undiscovered' for node in graph.nodes}

    status[source] = 'pending'
    pending = deque([source])

    # while there are still pending nodes
    while pending:
        u = pending.popleft()
        for v in graph.neighbors(u):
            # explore edge (u,v)
            if status[v] == 'undiscovered':
                status[v] = 'pending'
                # append to right
                pending.append(v)
        status[u] = 'visited'
```

Exercise

What is printed if we run a BFS starting at a?




```
...
while pending:
    u = pending.popleft()
    print(f'Popped {u}')
    for v in graph.neighbors(u):
        print(f'Exploring edge ({u}, {v})')
        # explore edge (u,v)
        ...
```

Answer

```
Popping a
Exploring edge (a, b)
Exploring edge (a, c)
Popping b
Exploring edge (b, a)
Exploring edge (b, c)
Exploring edge (b, d)
Popping c
Exploring edge (c, a)
Exploring edge (c, b)
Exploring edge (c, d)
Exploring edge (c, e)
```

```
Popping d
Exploring edge (d, b)
Exploring edge (d, c)
Exploring edge (d, e)
Popping e
Exploring edge (e, c)
Exploring edge (e, d)
Exploring edge (e, f)
Popping f
Exploring edge (f, e)
```

Aggregate Analysis

- During any one call to `bfs`:
  - Number of printed nodes: ?
  - Number of printed edges: ?

- In **aggregate** (over all calls):
  - Number of printed nodes: *exactly* $|V|$
  - Number of printed edges: *exactly* $2|E|$

Time Complexity

- Full BFS takes $\Theta(V + E)$

Time Complexity

- Full BFS takes $\Theta(V + E)$
- Why not just $\Theta(E)$?
- $\Theta(V + E)$ works *for all graphs*.
  - If we know more about the number of edges, we might be able to simplify.
  - E.g., if the graph is **complete**, $E = \Theta(V^2)$, so time complexity is $\Theta(V + V^2) = \Theta(V^2)$.

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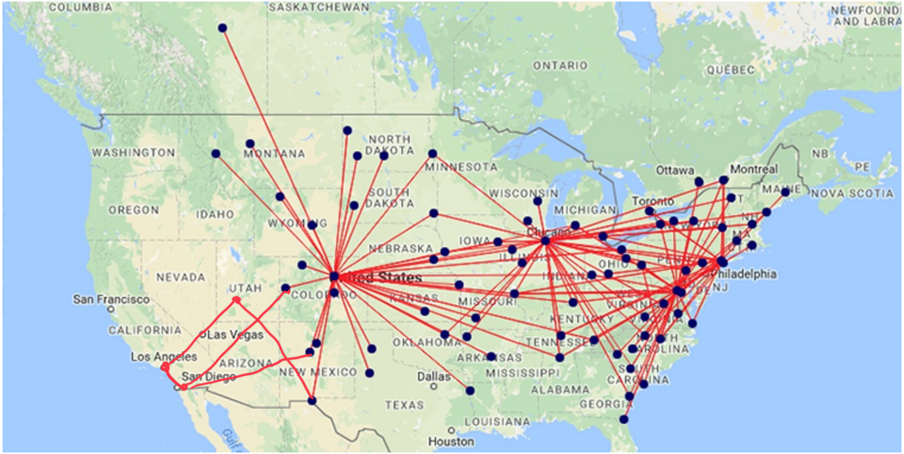
Lecture 12 | Part 2

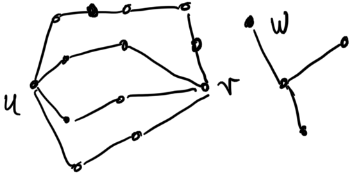
Shortest Paths


COLUMBIA
SASKATCHEWAN
ONTARIO
QUÉBEC
WASHINGTON
MONTANA
NORTH DAKOTA
MINNESOTA
OREGON
IDAHO
WYOMING
SOUTH DAKOTA
WISCONSIN
MICHIGAN
Ottawa
Montreal
Toronto
MAINE
NB
PE
NOVA SCOTIA
NEVADA
UTAH
NEBRASKA
IOWA
Chicago
OHIO
United States
Philadelphia
NJ
San Francisco
CALIFORNIA
COLORADO
KANSAS
MISSOURI
KENTUCKY
Las Vegas
OKLAHOMA
TENNESSEE
Los Angeles
ARIZONA
ARKANSAS
San Diego
NEW MEXICO
Dallas
MISSISSIPPI
ALABAMA
GEORGIA
TEXAS
LOUISIANA
Houston

Recall

- The **length** of a path is

$$(\#\text{ of nodes}) - 1$$



Definitions

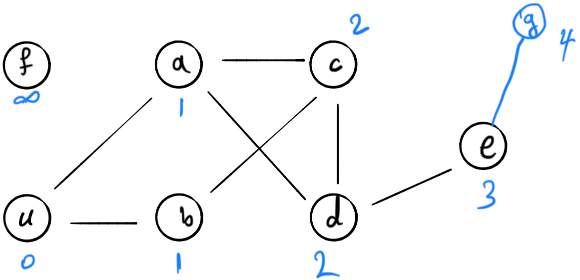
- A **shortest path** between *u* and *v* is a path between *u* and *v* with smallest possible length.
  - There may be several, or none at all.

- The **shortest path distance** is the length of a shortest path.
  - Convention: ∞ if no path exists.
  - “the distance between *u* and *v*” means spd.

Today: Shortest Paths

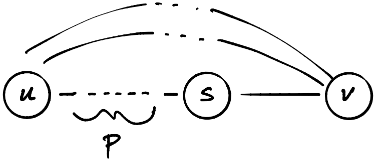
- **Given**: directed/undirected graph G, source u
- **Goal**: find shortest path from u to every other node

Example

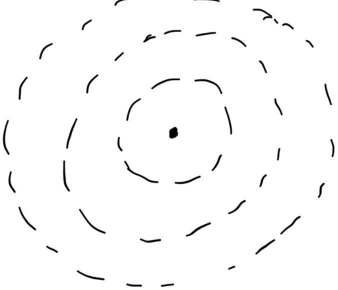


Key Property

- A shortest path of length k is composed of:
  - A **shortest path** of length $k - 1$.
  - Plus one edge.



Algorithm Idea



- Find all nodes distance 1 from source.
- Use these to find all nodes distance 2 from source.
- Use these to find all nodes distance 3 from source.
- ...

It turns out...

...this is exactly what BFS does.

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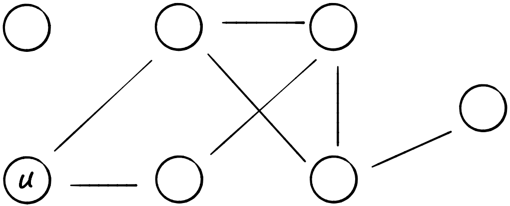
Lecture 12 | Part 3

BFS for Shortest Paths

Key Property of BFS

- For any $k \geq 1$ you choose:
- All nodes distance $k - 1$ from source are added to the queue before any node of distance k.
- Therefore, nodes are “processed” (popped from queue) in order of distance from source.

Example



Discovering Shortest Paths

- We “discover” shortest paths when we pop a node from queue and look at its neighbors.
- But the neighbor’s status matters!

Consider This

- We pop a node s.
- It has a neighbor v whose status is **undiscovered**.
- We've discovered a **shortest path** to v through s!

Consider This

- We pop a node s.
- It has a neighbor v whose status is **pending** or **visited**.
- We already have a shortest path to v.

Modifying BFS

- Use BFS “framework”.
- Return dictionary of **search predecessors**.
  - If v is discovered while visiting u, we say that u is the **BFS predecessor** of v.
  - This encodes the shortest paths.
- Also return dictionary of shortest path distances.

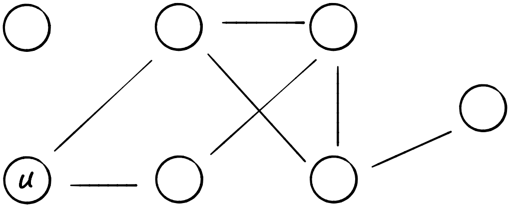
```python
def bfs_shortest_paths(graph, source):
    """Start a BFS at `source`."""
    status = {node: 'undiscovered' for node in graph.nodes}
    distance = {node: float('inf') for node in graph.nodes}
    predecessor = {node: None for node in graph.nodes}

    status[source] = 'pending'
    distance[source] = 0
    pending = deque([source])

    # while there are still pending nodes
    while pending:
        u = pending.popleft()
        for v in graph.neighbors(u):
            # explore edge (u,v)
            if status[v] == 'undiscovered':
                status[v] = 'pending'
                distance[v] = distance[u] + 1
                predecessor[v] = u
                # append to right
                pending.append(v)
        status[u] = 'visited'

    return predecessor, distance
```

Example



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Lecture 12 | Part 4

BFS Trees

Result of BFS

- Each node reachable from source has a single BFS predecessor.
  - Except for the source itself.

- The result is a **tree** (or forest).

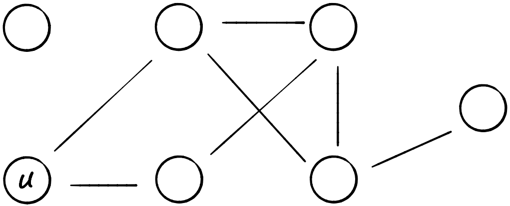
Trees

- A (free) **tree** is an undirected graph $T = (V, E)$ such that T is connected and $|E| = |V| - 1$.
- A **forest** is graph in which each connected component is a tree.

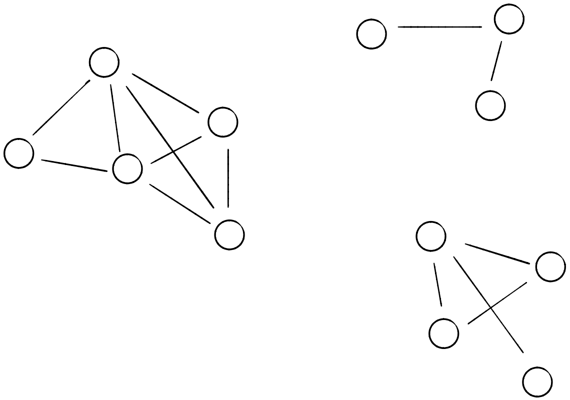
BFS Trees (Forests)

- If the input is connected, BFS produces a **tree**.
- If the input is not connected, BFS produces a **forest**.

Example



Example



BFS Trees

- BFS trees and forests encode shortest path distances.