

# DSC 40B

## *Theoretical Foundations II*

Lecture 7 | Part 1

## **The Median and Order Statistics**

# The Median

- ▶ How fast can we find a **median** of  $n$  numbers?

# Algorithms

- ▶ We have seen several ways of computing a median:
  - ▶ Alg. 1: Minimize absolute error, brute force.
  - ▶ Alg. 2: Use definition (half  $\leq$ , half  $\geq$ ).
  - ▶ ...

## Exercise

Using what we know so far, what approach for finding the median has the best **worst-case time complexity**?

## Best so far...

- ▶ Sort the list with mergesort, return middle element.
- ▶ Time complexity:  $\Theta(n \log n)$ .

# Is sorting necessary?

- ▶ Need to sort the whole list just to find middle?
- ▶ Seems like more work than necessary.

# Today

- ▶ We'll design an algorithm which runs in  $\Theta(n)$  expected time.
- ▶ Much more useful than just finding median...

# Order Statistics

- ▶ The median is an example of an **order statistic**.

## Definition

Given  $n$  numbers, the  **$k$ th order statistic** is the  $k$ th smallest number in the collection.



# Example

[99, 42, -77, -12, 101]

- ▶ 1st order statistic:
- ▶ 2nd order statistic:
- ▶ 4th order statistic:

## Exercise

Some special cases of order statistics go by different names. Can you think of some?

# Special Cases

- ▶ **Minimum:** 1st order statistic.
- ▶ **Maximum:**  $n$ th order statistic.
- ▶ **Median:**  $\lceil n/2 \rceil$ th order statistic<sup>1</sup>.
- ▶  **$p$ th Percentile:**  $\lceil \frac{p}{100} \cdot n \rceil$ th order statistic.

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<sup>1</sup>What if  $n$  is even?

# Goal

- ▶ **Fast** algorithm for computing any order statistic.
- ▶ Interestingly, some seem easier than others.
- ▶ Our algorithm will find **any** order statistic in  $\Theta(n)$  *expected* time.

# Approach #1

- ▶ We can modify `selection_sort` to find the  $k$ th order statistic.
- ▶ Loop invariant: after  $k$ th iteration, first  $k$  elements are in final sorted order.

```
def selection_sort(arr):  
    """In-place selection sort."""  
    n = len(arr)  
    if n <= 1:  
        return  
    for barrier_ix in range(n-1):  
        # find index of min in arr[start:]  
        min_ix = find_minimum(arr, start=barrier_ix)  
        #swap  
        arr[barrier_ix], arr[min_ix] = (  
            arr[min_ix], arr[barrier_ix]  
        )
```

```
def select_k(arr, k):  
    """Find kth order statistic."""  
    n = len(arr)  
    if n <= 1:  
        return  
    for barrier_ix in range(k):  
        # find index of min in arr[start:]  
        min_ix = find_minimum(arr, start=barrier_ix)  
        #swap  
        arr[barrier_ix], arr[min_ix] = (  
            arr[min_ix], arr[barrier_ix]  
        )  
    return arr[k-1]
```

## Exercise

What are the best case and worst case time complexities of `select_k`?



# Approach #1

- ▶ 1st order statistic:  $\Theta(n)$ .
- ▶  $n$ th order statistic:  $\Theta(n^2)$ .
- ▶ Median:  $\Theta(n^2)$ .
- ▶  $k$ th order statistic:  $\Theta(kn)$ .

## Exercise

Describe how to find any order statistic in  $\Theta(n \log n)$  time.

## Approach #2

- ▶ Sort with mergesort, return `arr[k-1]`
- ▶  $\Theta(n \log n)$  time. Could be better...

# DSC 40B

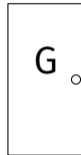
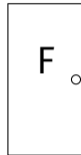
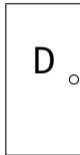
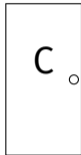
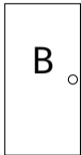
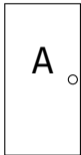
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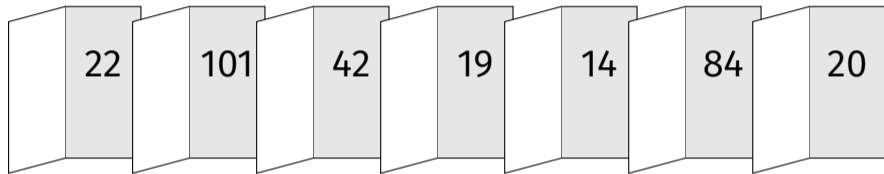
Lecture 7 | Part 2

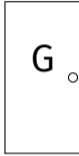
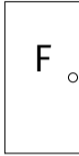
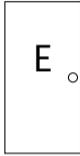
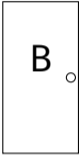
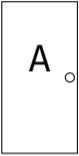
**Quickselect**

# The Goal

- ▶ Given a collection of  $n$  numbers and an order,  $k$ .
- ▶ Find the  $k$ th smallest number in the collection.



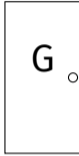
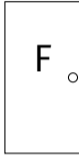
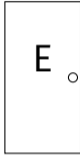
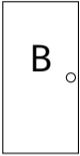
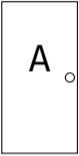


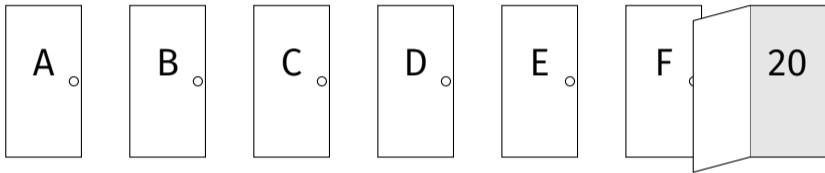


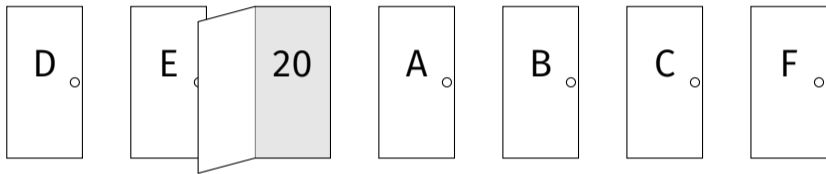


# Game Show

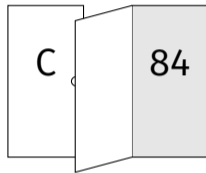
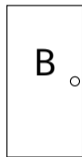
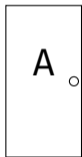
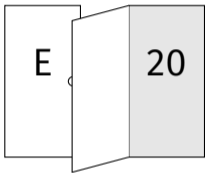
- ▶ **Goal:** tell the host the **largest** number.
- ▶ **Caution:** with every door opened, your money is reduced.
- ▶ **Twist:** After opening a door, the host tells you:
  - ▶ which doors are smaller.
  - ▶ which doors are larger.
  - ▶ they **partition** the doors into higher and lower by moving them.

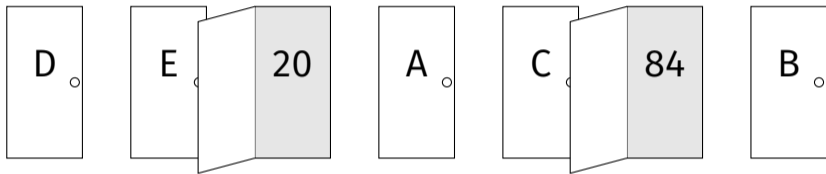




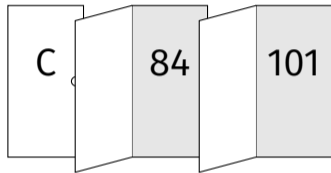
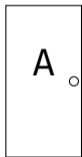
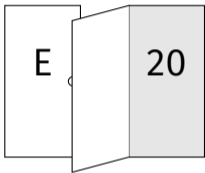


after partitioning





after partitioning



## Main Idea

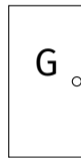
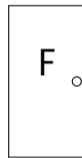
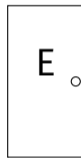
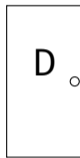
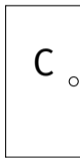
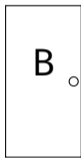
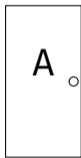
After partitioning, the just-opened door is in the **correct place** in the sorted order (but the other doors may not be).

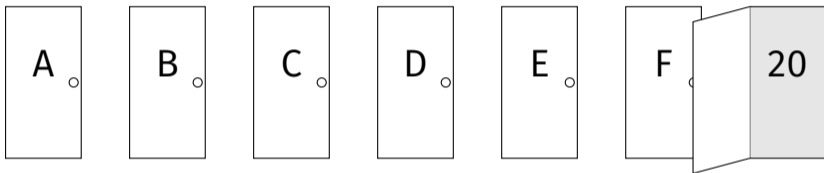
But, every door to the left is smaller ( $\leq$ ), every door to the right is larger ( $\geq$ ).

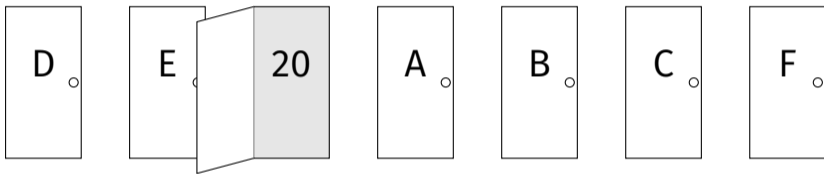


## In general...

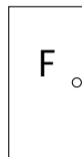
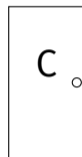
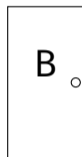
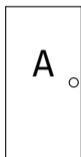
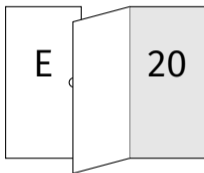
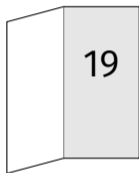
- ▶ Let's generalize strategy for  $k$ th order statistic.
- ▶ Example:  $k = 2$ .

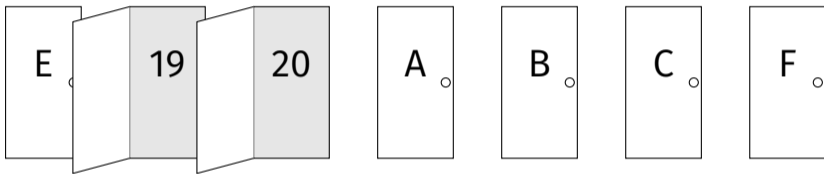






after partitioning





after partitioning

# Strategy

- ▶ Open arbitrary door (that hasn't been ruled out).
- ▶ **Partition** the doors around this number:
  - ▶ Move doors smaller than this to the left,
  - ▶ Larger than this to the right.
- ▶ Let  $p$  be our door's new position,  $k$  be the order we want.
  - ▶ If  $p = k$ , return this door.
  - ▶ If  $p < k$ , rule out doors to left.
  - ▶ If  $p > k$ , rule out doors to right.
- ▶ Repeat.

# In Code

```
import random
def quickselect(arr, k, start, stop):
    """Finds kth order statistic in numbers[start:stop)"""
    pivot_ix = random.randrange(start, stop)
    pivot_ix = partition(arr, start, stop, pivot_ix)
    pivot_order = pivot_ix + 1
    if pivot_order == k:
        return arr[pivot_ix]
    elif pivot_order < k:
        return quickselect(arr, k, pivot_ix + 1, stop)
    else:
        return quickselect(arr, k, start, pivot_ix)
```



# Example

`arr = [77, 42, 11, 99, 0, 101]`      `k = 3`

# DSC 40B

## *Theoretical Foundations II*

Lecture 7 | Part 3

**Partition**

# Partitioning

- ▶ Given an array of  $n$  numbers and the index of a **pivot**  $p$ .
- ▶ Rearrange elements so that:
  - ▶ Everything  $< p$  is first.
  - ▶ Everything  $= p$  is next.
  - ▶ Everything  $> p$  is last.
- ▶ Return index of first element  $\geq p$ .

```
def partition(arr, start, stop, pivot_ix):
    """Partition arr[start:stop] around pivot."""
    left = []
    pivot_count = 0
    right = []
    pivot = arr[pivot_ix]
    for ix in range(start, stop):
        if arr[ix] < pivot:
            left.append(arr[ix])
        elif arr[ix] == pivot:
            pivot_count += 1
        else:
            right.append(arr[ix])
    ix = start
    for x in left:
        arr[ix] = x
        ix += 1
    for i in range(pivot_count):
        arr[ix] = pivot
        ix += 1
    for x in right:
        arr[ix] = x
        ix += 1
    return start + len(left)
```

# Partition

- ▶ partition takes  $\Theta(n)$  time.
  - ▶ This is **optimal**.
- ▶ But we can use memory more efficiently.

# Motivation

- ▶ Similar to selection sort, we'll use **two** barriers:
- ▶ “Middle” barrier:
  - ▶ Separates things  $<$  pivot from things  $\geq$
  - ▶ Points to first thing in “right”
- ▶ “End” barrier:
  - ▶ Separates processed from processed.
  - ▶ Points to first unprocessed thing.

# Example

Simplification: start by moving pivot to end.

```
arr = [77, 42, 11, 99, 0, 101]    pivot = 1
```

```
def in_place_partition(arr, start, stop, pivot_ix):
    def swap(ix_1, ix_2):
        arr[ix_1], arr[ix_2] = arr[ix_2], arr[ix_1]

    pivot = arr[pivot_ix]
    swap(pivot_ix, stop-1)
    middle_barrier = start
    for end_barrier in range(start, stop - 1):
        if arr[end_barrier] < pivot:
            swap(middle_barrier, end_barrier)
            middle_barrier += 1
        # else:
        #     do nothing
    swap(middle_barrier, stop-1)
    return middle_barrier
```



# Efficiency

- ▶ Also takes  $\Theta(n)$  time.
- ▶ No auxiliary memory required.

# DSC 40B

## *Theoretical Foundations II*

Lecture 7 | Part 4

### **Time Complexity Analysis**

# Time Complexity

- ▶ What is time complexity of quickselect?

```
import random
def quickselect(arr, k, start, stop):
    """Finds kth order statistic in numbers[start:stop)"""
    pivot_ix = random.randrange(start, stop)
    pivot_ix = partition(arr, start, stop, pivot_ix)
    pivot_order = pivot_ix + 1
    if pivot_order == k:
        return arr[pivot_ix]
    elif pivot_order < k:
        return quickselect(arr, k, pivot_ix + 1, stop)
    else:
```

# Problem

- ▶ We don't know the size of the subproblem.
  - ▶ Is random, can be anywhere from 1 to  $n - 1$ .
- ▶ Difficult to write recurrence relation.

# Good and Bad Pivots

- ▶ Some pivots are better than others.
  - ▶ **Good**: splits array into roughly balanced halves.
  - ▶ **Bad**: splits array into wildly unbalanced pieces.

## Exercise

Suppose we're searching for the minimum. What would be the worst possible pivot?

# Worst Case

- ▶ Suppose we're searching for  $k = 1$  (minimum).
- ▶ Worst pivot: the maximum.
- ▶ Worst case: use max as pivot every time.
- ▶ Subproblem size:  $n - 1$ .



# Worst Case

- ▶ Every recursive call is on problem of size  $n - 1$ .
- ▶  $T(n) = T(n - 1) + \Theta(n)$ .
  - ▶ Solution:  $\Theta(n^2)$ .
- ▶ Intuitively, randomly choosing largest number as pivot every time is **very** unlikely!

$$\frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2} \times \dots \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{n!}$$

# Equally Unlikely

- ▶ Pivot falls exactly in the middle, every time.
- ▶ Subproblems are of size  $n/2$ .
- ▶  $T(n) = T(n/2) + \Theta(n)$ .
  - ▶ Solution:  $\Theta(n)$ .

# Typically

- ▶ Pivot falls somewhere in the middle.
- ▶ Sometimes **good**, sometimes **bad**.
- ▶ But **good** pivots reduce problem size by **so much** that they make up for **bad** pivots.

# Analogy

- ▶ You're 100 miles away from home.
- ▶ You have a button that, if you press it, teleports you **1 mile** closer to home.
- ▶ How many times must you press it before you're 1 mile away from home?

# Analogy

- ▶ You're 100 miles away from home.
- ▶ You have a button that, if you press it, teleports you **half the distance** to home.
- ▶ How many times must you press it before you're 1 mile away from home?

# Analogy

- ▶ You're 100 miles away from home.
- ▶ You have a button that, if you press it, teleports you **half the distance** to home with probability  $1/2$ , does nothing with probability  $1/2$ .
- ▶ How many times must you press it before you're 1 mile away from home?

# Analysis

- ▶ We'll call a pivot **good** if it falls in  $[\frac{n}{4}, \frac{3n}{4}]$ .
  - ▶ Probability:  $1/2$
  - ▶ Max problem size:  $3n/4$ .
  
- ▶ We'll call a pivot **bad** if it falls outside  $[\frac{n}{4}, \frac{3n}{4}]$ .
  - ▶ Probability:  $1/2$
  - ▶ Max problem size:  $n - 1$ .

# Observation

$T(n)$  = time to get from  $n$  to base case



# Observation

$$\begin{aligned} T(n) = & \text{time to get from } n \text{ to } \frac{3}{4}n \\ & + \text{time to get from } \frac{3}{4}n \text{ to } \left(\frac{3}{4}\right)^2 n \\ & + \text{time to get from } \left(\frac{3}{4}\right)^2 n \text{ to } \left(\frac{3}{4}\right)^3 n \\ & + \dots \end{aligned}$$

# Observation

Expected  $T(n)$  = expected time to get from  $n$  to  $\frac{3}{4}n$   
+ expected time to get from  $\frac{3}{4}n$  to  $\left(\frac{3}{4}\right)^2 n$   
+ expected time to get from  $\left(\frac{3}{4}\right)^2 n$  to  $\left(\frac{3}{4}\right)^3 n$   
+ ...

## Related

- ▶ What is the expected number of coin flips necessary in order to see “heads”?

## Related

- ▶ What is the expected number of coin flips necessary in order to see “heads”?
- ▶ Answer: 2

# Implication

- ▶ Expected number of calls necessary to go from  $n$  to  $3n/4$  is two.
- ▶ First call does  $cn$  work, second does  $c \times (3/4)n$ , third does  $c \times (3/4)^2n, \dots$

# Observation

Expected  $T(n)$  = expected time to get from  $n$  to  $\frac{3}{4}n$   
+ expected time to get from  $\frac{3}{4}n$  to  $\left(\frac{3}{4}\right)^2 n$   
+ expected time to get from  $\left(\frac{3}{4}\right)^2 n$  to  $\left(\frac{3}{4}\right)^3 n$   
+ ...

# Total Expected Time

$$2cn + 2\left(\frac{3}{4}\right)cn + 2\left(\frac{3}{4}\right)^2 cn + \dots = 2cn \cdot \sum_{p=0}^{\infty} \left(\frac{3}{4}\right)^p$$

# Quickselect

- ▶ Expected time complexity:  $\Theta(n)$ .
- ▶ Worst case:  $\Theta(n^2)$ , but **very unlikely**.



# Median

- ▶ We can find the median in expected linear time with **quickselect**.

# DSC 40B

## *Theoretical Foundations II*

Lecture 7 | Part 5

**Quicksort**

# Last Time

- ▶ We saw mergesort.
- ▶ **Divide:** split list directly down the middle
- ▶ **Conquer:** sort each half
- ▶ **Combine:** merge sorted halves together

## merge does all the work

- ▶ In mergesort, we are lazy when we divide.
- ▶ So we have to work to combine.

$[4, 1, 3, 2] \rightarrow [4, 1], [3, 2] \rightarrow [4, 3], [2, 3] \rightarrow [1, 2, 3, 4]$

# What if?

- ▶ Suppose we divide so that everything in left is smaller than everything in right:
- ▶ After sorting, no need for merge.
- ▶  $[5, 1, 3, 8, 6, 2] \rightarrow [1, 3, 2], [5, 8, 6]$

# What if?

- ▶ Suppose we divide so that everything in left is smaller than everything in right:
- ▶ After sorting, no need for merge.
- ▶  $[5, 1, 3, 8, 6, 2] \rightarrow [1, 3, 2], [5, 8, 6]$
- ▶ This is what partition does!

# Quicksort

```
def quicksort(arr, start, stop):  
    """Sort arr[start:stop] in-place."""  
    if stop - start > 1:  
        pivot_ix = random.randrange(start, stop)  
        pivot_ix = partition(arr, start, stop, pivot_ix)  
        quicksort(arr, start, pivot_ix)  
        quicksort(arr, pivot_ix+1, stop)
```

# Time Complexity

- ▶ Average case:  $\Theta(n \log n)$
- ▶ Worst case:  $\Theta(n^2)$ .
- ▶ Like with quickselect, worst case is **very rare**.



# Mergesort vs Quicksort

- ▶ Mergesort has better worst case complexity.
- ▶ But in practice, Quicksort is often faster.
- ▶ Takes less memory, too.

# Memory Requirements

- ▶ merges requires output array,  $\Theta(n)$  additional space.
- ▶ partition works in-place, requires no additional space<sup>2</sup>
- ▶ Example: sorting 3 GB of data with 4 GB of RAM.

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<sup>2</sup>Call stack for quicksort requires  $\Theta(\log n)$  additional space.